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Spherical functions on the de Sitter group

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Abstract
Matrix elements and spherical functions of irreducible representations of the de Sitter group are studied on the various homogeneous spaces of this group. It is shown that a universal covering of the de Sitter group gives rise to quaternion Euler angles. An explicit form of Casimir and Laplace–Beltrami operators on the homogeneous spaces is given. Different expressions of the matrix elements and spherical functions are given in terms of multiple hypergeometric functions both for finite-dimensional and unitary representations of the principal series of the de Sitter group. Applications of the functions obtained to the hydrogen atom problem are considered.

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1. Introduction

The representation theory of the de Sitter group, and also all the questions concerning this group and the de Sitter spacetime, is at the forefront due to the recent discoveries in modern cosmology. One of the most important problems in this area is a construction of quantum field theory in the de Sitter spacetime (see, for example, [2, 7, 19, 29]). As is known, in the standard quantum field theory in Minkowski spacetime solutions (wavefunctions) of relativistic wave equations are expressed via an expansion in relativistic spherical functions (matrix elements of the Lorentz group representations) [1, 24, 28, 30]. The analogous problem in five dimensions (solutions of wave equations in de Sitter space) requires the most exact definition for the matrix elements and spherical functions of irreducible representations of the de Sitter group.

In the present work, spherical functions are studied on the various homogeneous spaces of the de Sitter group $SO_0(1,4)$. A starting point of this research is an analogue between universal coverings of the Lorentz and de Sitter groups, which was first established by Takahashi [23] (see also the work of Ström [22]). Namely, the universal covering of $SO_0(1,4)$ is $\mathbf{Spin}_+(1,4) \simeq Sp(1,1)$ and the spinor group $\mathbf{Spin}_+(1,4)$ is described in terms of 2×2 quaternionic matrices. On the other hand, the universal covering of the Lorentz group $SO_0(1,3)$ is $\mathbf{Spin}_+(1,3) \simeq SL(2,\mathbb{C})$, where the spinor group $\mathbf{Spin}_+(1,3)$ is described in

terms of 2×2 complex matrices. This analogue allows us to apply (with some restrictions) Gel'fand–Naimark representation theory of the Lorentz group [15, 20] to $SO_0(1,4)$. Section 2 contains a further development of the Takahashi–Ström analogue (quaternionic description of $SO_0(1,4)$). It is shown that for the group $\mathbf{Spin}_+(1,4) \simeq Sp(1,1)$ there are quaternion Euler angles which contain complex Euler angles of $\mathbf{Spin}_+(1,3) \simeq SL(2,\mathbb{C})$ as a particular case. Differential operators (Laplace–Beltrami and Casimir operators) are defined on $Sp(1,1)$ in terms of the quaternion Euler angles. Spherical functions on the group $SO_0(1,4)$ are understood as functions of representations of the class 1 realized on the homogeneous spaces of $SO_0(1,4)$. A list of homogeneous spaces of $SO_0(1,4)$, including symmetric Riemannian and non-Riemannian spaces, is given at the end of section 2. Spherical functions on the group $SO(4)$ (maximal compact subgroup of $SO_0(1,4)$) are studied in section 3. It is shown that for a universal covering $\mathbf{Spin}(4) \simeq SU(2) \otimes SU(2)$ of $SO(4)$ there are double Euler angles. It should be noted that all the hypercomplex extensions (complex, double, quaternion) of usual Euler angles of the group $SU(2)$ follow directly from the algebraic structure underlying the groups $\mathbf{Spin}_+(p,q)$ and describing within the framework of Clifford algebras $C\ell_{p,q}$ [27]. Matrix elements and spherical functions of $SO(4)$ are expressed via the product of two hypergeometric functions. Further, spherical functions of finite-dimensional representations of $SO_0(1,4)$ are studied in section 4 on the various homogeneous spaces of $SO_0(1,4)$. It is shown that matrix elements of $SO_0(1,4)$ admit factorizations with respect to the matrix elements of subgroups $SO(4)$ and $SO_0(1,3)$, since double and complex angles are particular cases of the quaternion angles. In turn, matrix elements and spherical functions of $SO_0(1,4)$ are expressed via multiple hypergeometric series (the product of three hypergeometric functions). At the end of section 4 we consider applications of the spherical functions, defined on the four-dimensional hyperboloid, to hydrogen and antihydrogen atom problems. Spherical functions of the principal series representations of $SO_0(1,4)$ are considered in section 5 within the Dixmier–Ström representation basis of the de Sitter group $SO_0(1,4)$ [9, 22].

2. The de Sitter group $SO_0(1,4)$

The homogeneous de Sitter group $SO_0(1,4)$ consists of all real matrices of fifth order with the unit determinant which leave invariant the quadratic form

$$Q(x) = x_0^2 - x_1^2 - x_2^2 - x_3^2 - x_4^2.$$

The Lie algebra $\mathfrak{so}(1,4)$ of $SO_0(1,4)$ consists of all real matrices

$$\begin{bmatrix} 0 & a_{01} & a_{02} & a_{03} & a_{04} \\ a_{01} & 0 & -a_{12} & -a_{13} & -a_{14} \\ a_{02} & a_{12} & 0 & -a_{23} & -a_{24} \\ a_{03} & a_{13} & a_{23} & 0 & -a_{34} \\ a_{04} & a_{14} & a_{24} & a_{34} & 0 \end{bmatrix}. \tag{1}$$

Thus, the algebra $\mathfrak{so}(1,4)$ has basis elements of the form

$$L_{rs} = -e_{rs} + e_{sr}, \qquad s, r = 1, 2, 3, 4, \quad s < r, \tag{2}$$

$$L_{0r} = e_{0r} + e_{r0}, \qquad r = 1, 2, 3, 4, \tag{3}$$

where e_{rs} is a matrix with elements $(e_{rs})_{pq} = \delta_{rp}\delta_{sq}$. The basis elements (2) and (3) satisfy the following commutation relations:

$$\begin{aligned} &[L_{\mu\nu}, L_{\rho\sigma}] = g_{\nu\rho}L_{\mu\sigma} + g_{\mu\sigma}L_{\nu\rho} - g_{\mu\rho}L_{\nu\sigma} - g_{\nu\sigma}L_{\mu\rho}, \\ &\rho, \mu, \nu, \sigma = 0, 1, 2, 3, 4, \end{aligned} \tag{4}$$

where $g_{k0} = g_{0k} = \delta_{0k}$, $g_{ks} = -\delta_{ks}$; $k, s = 1, 2, 3, 4$. $SO_0(1,4)$ is a ten-parametric group.

The maximal compact subgroup K of $SO_0(1,4)$ is isomorphic to the group $SO(4)$ and consists of the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & SO(4) \end{pmatrix}.$$

Further, Cartan decomposition of the algebra $\mathfrak{so}(1,4)$ and Iwasawa decomposition of the group $SO_0(1,4)$ have a great importance at the construction of representations of the de Sitter group $SO_0(1,4)$. So, in the Cartan decomposition $\mathfrak{so}(1,4) = \mathfrak{so}(4) + \mathfrak{p}$ a subspace $\mathfrak{p}$ consists of the basis elements (3). The group $SO_0(1,4)$ has a real rank 1. For that reason, the commutative subalgebra $\mathfrak{a}$ of $\mathfrak{so}(1,4)$ is one dimensional. We can take the matrix L_{04} as a basis element of $\mathfrak{a}$. Therefore, the commutative subgroup A consists of the matrices

$$\begin{bmatrix} \cosh\alpha & 0 & 0 & 0 & \sinh\alpha \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \sinh\alpha & 0 & 0 & 0 & \cosh\alpha \end{bmatrix}, \qquad 0 \leqslant \alpha \leqslant \infty. \tag{5}$$

Using relations (4), we verify that a nilpotent subalgebra $\mathfrak{n}$ of $\mathfrak{so}(1,4)$ is defined by the matrices $L_{02} + L_{24}$, $L_{03} + L_{34}$ and $L_{01} + L_{14}$. Making an exponential mapping of the subalgebra $\mathfrak{n}$ into the subgroup N, we find that the nilpotent subgroup N consists of the matrices

$$\begin{bmatrix} 1 + (r^2+s^2+t^2)/2 & t & r & s & -(r^2+s^2+t^2) \\ t & 1 & 0 & 0 & -t \\ r & 0 & 1 & 0 & -r \\ s & 0 & 0 & 1 & -s \\ (r^2+s^2+t^2)/2 & t & r & s & 1-(r^2+s^2+t^2) \end{bmatrix}. \tag{6}$$

The subgroups K, A and N define the Iwasawa decomposition $SO_0(1,4) = SO(4) \cdot NA$. In accordance with the definition of the subgroup M of $SO_0(1,4)$ (see, for example, [18]), the subgroup M is isomorphic to $SO(3)$. Thus, a minimal parabolic subgroup P has a decomposition $P = SO(3) \cdot NA$. Since the rank of $SO_0(1,4)$ is equal to 1, then there exist no other parabolic subgroups containing P.

In the group $SO_0(1,4)$ there are two independent Casimir operators

$$F = L_{12}^2 + L_{13}^2 + L_{14}^2 + L_{23}^2 + L_{24}^2 + L_{34}^2 - L_{01}^2 - L_{02}^2 - L_{03}^2 - L_{04}^2, \tag{7}$$

$$\begin{aligned} W = {} & (L_{12}L_{24} - L_{13}L_{24} + L_{14}L_{23})^2 - (L_{12}L_{34} - L_{03}L_{24} + L_{04}L_{23})^2 \\ & - (L_{01}L_{34} - L_{03}L_{14} + L_{04}L_{13})^2 - (L_{01}L_{24} - L_{02}L_{14} + L_{04}L_{12})^2 \\ & - (L_{01}L_{23} - L_{02}L_{13} + L_{03}L_{12})^2. \end{aligned} \tag{8}$$

It is known that Casimir operator W is equal to zero on the representations T^σ of the class 1 [6]. The Casimir operator F takes the values $\sigma(\sigma+3)$ on the representations T^σ.

With the aim to obtain self-conjugated operators we will consider generators $J_{\mu\nu} = \mathbf{i}L_{\mu\nu}$ instead the elements $L_{\mu\nu}$ of the algebra $\mathfrak{so}(1,4)$. In unitary representations we have $J_{\mu\nu}^* = J_{\mu\nu}$. Let us introduce the following designations for the ten generators $J_{\mu\nu}$ of $SO_0(1,4)$:

$$\begin{aligned} & \boldsymbol{M} = (M_1 \equiv J_{23}, M_2 \equiv J_{31}, M_3 \equiv J_{12}), \\ & \boldsymbol{P} = (P_1 \equiv J_{14}, P_2 \equiv J_{24}, P_3 \equiv J_{34}), \\ & \boldsymbol{N} = (N_1 \equiv J_{01}, N_2 \equiv J_{02}, N_3 \equiv J_{03}), \\ & P_0 = J_{04}. \end{aligned} \tag{9}$$

Casimir operators of the group $SO_0(1,4)$ in this designation have the form

$$F = \left(P_0^2 + \boldsymbol{N}^2\right) - (\boldsymbol{P}^2 + \boldsymbol{M}^2),$$
$$W = (\boldsymbol{M}\cdot\boldsymbol{P})^2 - (P_0\boldsymbol{M} - \boldsymbol{P}\times\boldsymbol{N})^2 - (\boldsymbol{M}\cdot\boldsymbol{N})^2.$$

The generators (9) satisfy the following commutation relations:

$$\begin{aligned}
&[M_k, M_l] = \mathbf{i}\varepsilon_{klm}M_m, \qquad [N_k, N_l] = -\mathbf{i}\varepsilon_{klm}M_m,\\
&[P_k, P_l] = \mathbf{i}\varepsilon_{klm}M_m,\\
&[M_k, N_l] = \mathbf{i}\varepsilon_{klm}N_m, \qquad [M_k, P_l] = \mathbf{i}\varepsilon_{klm}P_m,\\
&[M_k, N_k] = [M_k, P_k] = [M_k, P_0] = 0,\\
&[P_0, N_k] = \mathbf{i}P_k, \qquad [P_0, P_k] = \mathbf{i}N_k, \qquad [P_k, N_l] = \mathbf{i}\delta_{kl}P_0,
\end{aligned} \tag{10}$$

where ε_{klm} is an antisymmetric tensor of third rank, which takes the values 0 or ± 1 $(k, l, m = 1, 2, 3)$.

2.1. Quaternionic description of $SO_0(1,4)$

Universal covering of the de Sitter group $SO_0(1,4)$ is a spinor group $\mathbf{Spin}_+(1,4) \simeq Sp(1,1)$ [16, 27]. In its turn, $\mathbf{Spin}_+(1,4) \in C\ell^+_{1,4}$, where $C\ell^+_{1,4}$ is an even subalgebra of the Clifford algebra $C\ell_{1,4}$ associated with the de Sitter space $\mathbb{R}^{1,4}$. Further, there is an isomorphism $C\ell^+_{1,4} \simeq C\ell_{1,3}$, where $C\ell_{1,3}$ is a spacetime algebra associated with the Minkowski space $\mathbb{R}^{1,3}$.

In virtue of the Karoubi theorem [17], the spacetime algebra $C\ell_{1,3}$ admits the following decomposition[1]:

$$C\ell_{1,3} \simeq C\ell_{1,1} \otimes C\ell_{0,2}.$$

The decomposition $C\ell_{1,3} \simeq C\ell_{1,1} \otimes C\ell_{0,2}$ means that for the algebra $C\ell_{1,3}$ there exists a transition from the real coordinates to quaternion coordinates of the form $a + b\zeta_1 + c\zeta_2 + d\zeta_1\zeta_2$, where $\zeta_1 = \mathbf{e}_{123}$, $\zeta_2 = \mathbf{e}_{124}$. At this point, $\zeta_1^2 = \zeta_2^2 = (\zeta_1\zeta_2)^2 = -1$, $\mathbf{e}_1^2 = 1$, $\mathbf{e}_2^2 = \mathbf{e}_3^2 = \mathbf{e}_4^2 = -1$. It is easy to see that the units ζ_1 and ζ_2 form a basis of the quaternion algebra, since $\zeta_1 \sim \mathbf{i}$, $\zeta_2 \sim \mathbf{j}$, $\zeta_1\zeta_2 \sim \mathbf{k}$. Therefore, a general element

$$\mathcal{A}_{C\ell_{1,3}} = a^0\mathbf{e}_0 + \sum_{i=1}^{4} a^i\mathbf{e}_i + \sum_{i=1}^{4}\sum_{j=1}^{4} a^{ij}\mathbf{e}_i\mathbf{e}_j + \sum_{i=1}^{4}\sum_{j=1}^{4}\sum_{k=1}^{4} a^{ijk}\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k + a^{1234}\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_4$$

of the spacetime algebra $C\ell_{1,3}$ can be written in the form

$$\mathcal{A}_{C\ell_{1,3}} = C\ell^0_{1,1} + C\ell^1_{1,1}\zeta_1 + C\ell^2_{1,1}\zeta_2 + C\ell^3_{1,1}\zeta_1\zeta_2,$$

where each coefficient $C\ell^i_{1,1}$ $(i = 0, 1, 2, 3)$ is isomorphic to the anti-quaternion algebra $C\ell_{1,1}$:[2]

$$\begin{aligned}
C\ell^0_{1,1} &= a^0 + a^1\mathbf{e}_1 + a^2\mathbf{e}_2 + a^{12}\mathbf{e}_{12},\\
C\ell^1_{1,1} &= a^{123} - a^{23}\mathbf{e}_1 - a^{13}\mathbf{e}_2 - a^3\mathbf{e}_{12},\\
C\ell^2_{1,1} &= a^{124} - a^{24}\mathbf{e}_1 + a^{14}\mathbf{e}_2 + a^4\mathbf{e}_{12},\\
C\ell^3_{1,1} &= -a^{34} - a^{134}\mathbf{e}_1 - a^{234}\mathbf{e}_2 + a^{1234}\mathbf{e}_{12}.
\end{aligned}$$

It is easy to verify that the units ζ_1 and ζ_2 commute with all the basis elements of $C\ell_{1,1}$.

[1] This decomposition is a particular case of the most general formula $C\ell(V\oplus V', Q\oplus Q') \simeq C\ell(V, Q)\otimes C\ell(V', -Q')$, where V and V' are vector spaces endowed with quadratic forms Q and Q' over the field $\mathbb{F}$, $\dim V$ is even [17, proposition 3.16].

[2] $C\ell_{1,1}$ is a real Clifford algebra of the type $p - q \equiv 0 \pmod 8$ with a division ring $\mathbb{K} \simeq \mathbb{R}$. This algebra is called the anti-quaternion algebra by Rozenfel'd [21].

Further, let us define matrix representations of the quaternion units ζ_1 and ζ_2 as follows:

$$\zeta_1 \longmapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \zeta_2 \longmapsto \begin{pmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix}.$$

Thus, in virtue of the Karoubi theorem we have

$$C\ell_{1,3} \simeq \mathrm{Mat}_2(C\ell_{1,1}) = \begin{bmatrix} C\ell^0_{1,1} - \mathbf{i}C\ell^3_{1,1} & -C\ell^1_{1,1} + C\ell^2_{1,1} \\ C\ell^1_{1,1} + \mathbf{i}C\ell^2_{1,1} & C\ell^0_{1,1} + \mathbf{i}C\ell^3_{1,1} \end{bmatrix},$$

or

$$C\ell_{1,3} \simeq \mathrm{Mat}_2(C\ell_{1,1}) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \begin{bmatrix} a^0 - a^{134} + (a^1 + a^{34})\mathbf{i} - (a^{13} + a^4)\mathbf{j} + (a^{14} - a^3)\mathbf{k} & a^{24} - a^{123} + (a^{23} + a^{124})\mathbf{i} + (a^{13} - a^4)\mathbf{j} + (a^{14} + a^3)\mathbf{k} \\ a^{24} + a^{123} + (a^{124} - a^{23})\mathbf{i} - (a^{13} + a^4)\mathbf{j} + (a^{14} - a^3)\mathbf{k} & a^0 + a^{134} + (a^1 - a^{34})\mathbf{i} + (a^2 - a^{1234})\mathbf{j} + (a^{12} + a^{234})\mathbf{k} \end{bmatrix},$$

where $\mathbf{i} = \mathbf{e}_1, \mathbf{j} = \mathbf{e}_2, \mathbf{k} = \mathbf{e}_{12}$ are anti-quaternion units, which satisfy the relations

$$\begin{aligned} &\mathbf{i}^2 = -1, && \mathbf{j}^2 = 1, && \mathbf{k}^2 = 1, \\ &\mathbf{ij} = -\mathbf{ji} = \mathbf{k}, && \mathbf{ki} = -\mathbf{ik} = \mathbf{j}, && \mathbf{kj} = -\mathbf{jk} = \mathbf{i}. \end{aligned}$$

In such a way, the universal covering of the de Sitter group $SO_0(1,4)$ is

$$\mathbf{Spin}_+(1,4) \simeq \left\{ \begin{bmatrix} a & b \\ c & c \end{bmatrix} \in \mathbb{H}(2) : \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = 1 \right\} = Sp(1,1),$$

where $\det \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] = 1$ means that

$$\overline{a}b = \overline{c}d, \qquad |a|^2 - |c|^2 = 1, \qquad |d|^2 - |b|^2 = 1,$$

or

$$a\overline{c} = b\overline{d}, \qquad |a|^2 - |b|^2 = 1, \qquad |d|^2 - |c|^2 = 1,$$

here $\overline{a}$ means a quaternion conjugation.

The ten-parameter group $\mathbf{Spin}_+(1,4) \simeq Sp(1,1)$ has the following one-parameter subgroups:

$$m_{12}(\psi) = \begin{pmatrix} \mathrm{e}^{\mathbf{i}\frac{\psi}{2}} & 0 \\ 0 & \mathrm{e}^{-\mathbf{i}\frac{\psi}{2}} \end{pmatrix}, \qquad m_{13}(\varphi) = \begin{pmatrix} \cos\frac{\varphi}{2} & -\sin\frac{\varphi}{2} \\ \sin\frac{\varphi}{2} & \cos\frac{\varphi}{2} \end{pmatrix}, \qquad m_{23}(\theta) = \begin{pmatrix} \cos\frac{\theta}{2} & \mathbf{i}\sin\frac{\theta}{2} \\ \mathbf{i}\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix},$$

$$p_{14}(\phi) = \begin{pmatrix} \cos\frac{\phi}{2} & \mathbf{i}\sin\frac{\phi}{2} \\ \mathbf{i}\sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{pmatrix}, \quad p_{24}(\varsigma) = \begin{pmatrix} \cos\frac{\varsigma}{2} & -\mathbf{j}\sin\frac{\varsigma}{2} \\ \mathbf{j}\sin\frac{\varsigma}{2} & \cos\frac{\varsigma}{2} \end{pmatrix}, \quad p_{34}(\chi) = \begin{pmatrix} \mathrm{e}^{\mathbf{k}\frac{\chi}{2}} & 0 \\ 0 & \mathrm{e}^{-\mathbf{k}\frac{\chi}{2}} \end{pmatrix},$$

$$n_{01}(\tau) = \begin{pmatrix} \cosh\frac{\tau}{2} & \sinh\frac{\tau}{2} \\ \sinh\frac{\tau}{2} & \cosh\frac{\tau}{2} \end{pmatrix}, \; n_{02}(\epsilon) = \begin{pmatrix} \cosh\frac{\epsilon}{2} & \mathbf{i}\sinh\frac{\epsilon}{2} \\ -\mathbf{i}\sinh\frac{\epsilon}{2} & \cosh\frac{\epsilon}{2} \end{pmatrix}, \; n_{03}(\varepsilon) = \begin{pmatrix} \mathrm{e}^{\frac{\varepsilon}{2}} & 0 \\ 0 & \mathrm{e}^{-\frac{\varepsilon}{2}} \end{pmatrix},$$

$$p_{04}(\omega) = \begin{pmatrix} \mathrm{e}^{\frac{\omega}{2}} & 0 \\ 0 & \mathrm{e}^{-\frac{\omega}{2}} \end{pmatrix},$$

where the ranges of parameters (Euler angles) are

$$\begin{aligned} &0 \leqslant \theta \leqslant \pi, && 0 \leqslant \phi \leqslant \pi, \\ &0 \leqslant \varphi < 2\pi, && 0 \leqslant \varsigma < 2\pi, \\ &-2\pi \leqslant \psi < 2\pi, && -2\pi \leqslant \chi < 2\pi, \end{aligned} \tag{11}$$

$$\begin{aligned} &-\infty < \tau < +\infty, \qquad -\infty < \epsilon < +\infty, \\ &-\infty < \varepsilon < +\infty, \qquad -\infty < \omega < +\infty. \end{aligned} \tag{12}$$

Let us find a general transformation $\mathfrak{q}$ of $\mathbf{Spin}_+(1,4)$ in the space of representation with the smallest weight (a so-called fundamental representation). In general, this form of the element $g \in G$ is closely related with the Cartan decomposition $G = KAK$, where G is a connected Lie group, K is a maximal compact subgroup of G and A is a maximal commutative subgroup of G. For example, the three-parameter group $SU(2)$ (a universal covering of $SO(3)$) has the following subgroups:

$$K = \left\{ \begin{pmatrix} \cos\frac{\theta}{2} & \mathbf{i}\sin\frac{\theta}{2} \\ \mathbf{i}\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \right\}, \qquad A = \left\{ \begin{pmatrix} \mathrm{e}^{\mathbf{i}\frac{t}{2}} & 0 \\ 0 & \mathrm{e}^{-\mathbf{i}\frac{t}{2}} \end{pmatrix} \right\}, \tag{13}$$

where $t = \{\varphi, \psi\}$. Therefore, the Cartan decomposition $SU(2) = KAK$ of the element $u \in SU(2)$ is (see, for example, [32])

$$g \equiv u(\varphi,\theta,\psi) = \begin{pmatrix} \mathrm{e}^{\mathbf{i}\frac{\varphi}{2}} & 0 \\ 0 & \mathrm{e}^{-\mathbf{i}\frac{\varphi}{2}} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} & \mathbf{i}\sin\frac{\theta}{2} \\ \mathbf{i}\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} \mathrm{e}^{\mathbf{i}\frac{\psi}{2}} & 0 \\ 0 & \mathrm{e}^{-\mathbf{i}\frac{\psi}{2}} \end{pmatrix}, \tag{14}$$

where φ, θ, ψ are Euler angles.

In its turn, the six-parameter group $\mathbf{Spin}_+(1,3) \simeq SL(2,\mathbb{C})$ (a universal covering of the Lorentz group $SO_0(1,3)$) is a complex extension of the group $SU(2)$, that is, $SL(2,\mathbb{C}) = [SU(2)]^c = K^c A^c K^c$, where K^c and A^c are complex extensions of the groups (13):

$$K^c = \left\{ \begin{pmatrix} \cos\frac{\theta^c}{2} & \mathbf{i}\sin\frac{\theta^c}{2} \\ \mathbf{i}\sin\frac{\theta^c}{2} & \cos\frac{\theta^c}{2} \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} & \mathbf{i}\sin\frac{\theta}{2} \\ \mathbf{i}\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} \cosh\frac{\tau}{2} & \sinh\frac{\tau}{2} \\ \sinh\frac{\tau}{2} & \cosh\frac{\tau}{2} \end{pmatrix} \right\},$$

$$A^c = \left\{ \begin{pmatrix} \mathrm{e}^{\mathbf{i}\frac{t^c}{2}} & 0 \\ 0 & \mathrm{e}^{-\mathbf{i}\frac{t^c}{2}} \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{\mathbf{i}\frac{p}{2}} & 0 \\ 0 & \mathrm{e}^{-\mathbf{i}\frac{p}{2}} \end{pmatrix} \begin{pmatrix} \mathrm{e}^{\frac{q}{2}} & 0 \\ 0 & \mathrm{e}^{-\frac{q}{2}} \end{pmatrix} \right\},$$

where $p = \{\varphi, \psi\}, q = \{\epsilon, \varepsilon\}$. Thus, the Cartan decomposition $SL(2,\mathbb{C}) = K^c A^c K^c$ of the element $\mathfrak{g} \in \mathbf{Spin}_+(1,3) \simeq SL(2,\mathbb{C})$ is

$$\begin{aligned} g &\equiv \mathfrak{g}(\varphi^c,\theta^c,\psi^c) = \mathfrak{g}(\varphi,\epsilon,\theta,\tau,\psi,\varepsilon) \\ &= \begin{pmatrix} \mathrm{e}^{\mathbf{i}\frac{\varphi}{2}} & 0 \\ 0 & \mathrm{e}^{-\mathbf{i}\frac{\varphi}{2}} \end{pmatrix} \begin{pmatrix} \mathrm{e}^{\frac{\epsilon}{2}} & 0 \\ 0 & \mathrm{e}^{-\frac{\epsilon}{2}} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} & \mathbf{i}\sin\frac{\theta}{2} \\ \mathbf{i}\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} \cosh\frac{\tau}{2} & \sinh\frac{\tau}{2} \\ \sinh\frac{\tau}{2} & \cosh\frac{\tau}{2} \end{pmatrix} \\ &\quad \times \begin{pmatrix} \mathrm{e}^{\mathbf{i}\frac{\psi}{2}} & 0 \\ 0 & \mathrm{e}^{-\mathbf{i}\frac{\psi}{2}} \end{pmatrix} \begin{pmatrix} \mathrm{e}^{\frac{\varepsilon}{2}} & 0 \\ 0 & \mathrm{e}^{-\frac{\varepsilon}{2}} \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{e}^{\mathbf{i}\frac{\varphi^c}{2}} & 0 \\ 0 & \mathrm{e}^{-\mathbf{i}\frac{\varphi^c}{2}} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta^c}{2} & \mathbf{i}\sin\frac{\theta^c}{2} \\ \mathbf{i}\sin\frac{\theta^c}{2} & \cos\frac{\theta^c}{2} \end{pmatrix} \begin{pmatrix} \mathrm{e}^{\mathbf{i}\frac{\psi^c}{2}} & 0 \\ 0 & \mathrm{e}^{-\mathbf{i}\frac{\psi^c}{2}} \end{pmatrix}, \end{aligned} \tag{15}$$

where

$$\varphi^c = \varphi - \mathbf{i}\epsilon, \qquad \theta^c = \theta - \mathbf{i}\tau, \qquad \psi^c = \psi - \mathbf{i}\varepsilon$$

are *complex Euler angles*. Hence it follows that the element (15) is a complex extension of (14).

Further, the six-parameter spinor group $\mathbf{Spin}(4)$ (a universal covering of $SO(4)$) due to an isomorphism $\mathbf{Spin}(4) \simeq SU(2) \otimes SU(2)$ admits the decomposition $\mathbf{Spin}(4) = K^e A^e K^e$,

where K^e and A^e are double extensions of the subgroups (13):

$$K^e = \left\{ \begin{pmatrix} \cos\frac{\theta^e}{2} & \mathbf{i}\sin\frac{\theta^e}{2} \\ \mathbf{i}\sin\frac{\theta^e}{2} & \cos\frac{\theta^e}{2} \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} & \mathbf{i}\sin\frac{\theta}{2} \\ \mathbf{i}\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} \cos\frac{\phi}{2} & \mathbf{i}\sin\frac{\phi}{2} \\ \mathbf{i}\sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{pmatrix} \right\},$$
$$A^e = \left\{ \begin{pmatrix} \mathrm{e}^{\mathbf{i}\frac{t^e}{2}} & 0 \\ 0 & \mathrm{e}^{-\mathbf{i}\frac{t^e}{2}} \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{\mathbf{i}\frac{p}{2}} & 0 \\ 0 & \mathrm{e}^{-\mathbf{i}\frac{p}{2}} \end{pmatrix} \begin{pmatrix} \mathrm{e}^{\mathbf{i}\frac{q}{2}} & 0 \\ 0 & \mathrm{e}^{-\mathbf{i}\frac{q}{2}} \end{pmatrix} \right\},$$

where $p = \{\varphi, \psi\}, q = \{\varsigma, \chi\}$. In this case, the Cartan decomposition $\mathbf{Spin}(4) = K^e A^e K^e$ of the element $g \in SU(2) \otimes SU(2)$ is

$$\begin{aligned} g \equiv g(\varphi^e, \theta^e, \psi^e) &= g(\varphi, \varsigma, \theta, \phi, \psi, \chi) \\ &= \begin{pmatrix} \mathrm{e}^{\mathbf{i}\frac{\varphi}{2}} & 0 \\ 0 & \mathrm{e}^{-\mathbf{i}\frac{\varphi}{2}} \end{pmatrix} \begin{pmatrix} \mathrm{e}^{\mathbf{i}\frac{\varsigma}{2}} & 0 \\ 0 & \mathrm{e}^{-\mathbf{i}\frac{\varsigma}{2}} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} & \mathbf{i}\sin\frac{\theta}{2} \\ \mathbf{i}\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} \cos\frac{\phi}{2} & \mathbf{i}\sin\frac{\phi}{2} \\ \mathbf{i}\sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{pmatrix} \\ &\quad \times \begin{pmatrix} \mathrm{e}^{\mathbf{i}\frac{\psi}{2}} & 0 \\ 0 & \mathrm{e}^{-\mathbf{i}\frac{\psi}{2}} \end{pmatrix} \begin{pmatrix} \mathrm{e}^{\mathbf{i}\frac{\chi}{2}} & 0 \\ 0 & \mathrm{e}^{-\mathbf{i}\frac{\chi}{2}} \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{e}^{\mathbf{i}\frac{\varphi^e}{2}} & 0 \\ 0 & \mathrm{e}^{-\mathbf{i}\frac{\varphi^e}{2}} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta^e}{2} & \mathbf{i}\sin\frac{\theta^e}{2} \\ \mathbf{i}\sin\frac{\theta^e}{2} & \cos\frac{\theta^e}{2} \end{pmatrix} \begin{pmatrix} \mathrm{e}^{\mathbf{i}\frac{\psi^e}{2}} & 0 \\ 0 & \mathrm{e}^{-\mathbf{i}\frac{\psi^e}{2}} \end{pmatrix}, \end{aligned} \tag{16}$$

where

$$\left.\begin{aligned} \theta^e &= \theta + \phi, \\ \varphi^e &= \varphi + \varsigma, \\ \psi^e &= \psi + \chi \end{aligned}\right\} \tag{17}$$

are *double Euler angles*. It is easy to see that the element (16) is a double extension of (14).

Finally, the ten-parameter spinor group $\mathbf{Spin}_+(1,4) \simeq Sp(1,1)$ (a universal covering of the de Sitter group $SO_0(1,4)$) is defined in terms of 2×2 quaternionic matrices. This fact allows us to introduce a decomposition $Sp(1,1) = K^q A^q K^q$, where K^q and A^q are quaternionic extensions of the groups (13):

$$K^q = \left\{ \begin{pmatrix} \cos\frac{\theta^q}{2} & \mathbf{i}\sin\frac{\theta^q}{2} \\ \mathbf{i}\sin\frac{\theta^q}{2} & \cos\frac{\theta^q}{2} \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} & \mathbf{i}\sin\frac{\theta}{2} \\ \mathbf{i}\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} \cosh\frac{\tau}{2} & \sinh\frac{\tau}{2} \\ \sinh\frac{\tau}{2} & \cosh\frac{\tau}{2} \end{pmatrix} \begin{pmatrix} \cos\frac{\phi}{2} & \mathbf{i}\sin\frac{\phi}{2} \\ \mathbf{i}\sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{pmatrix} \right\},$$
$$A^q = \left\{ \begin{pmatrix} \mathrm{e}^{\mathbf{i}\frac{\varphi^q}{2}} & 0 \\ 0 & \mathrm{e}^{-\mathbf{i}\frac{\varphi^q}{2}} \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{\mathbf{i}\frac{\varphi}{2}} & 0 \\ 0 & \mathrm{e}^{-\mathbf{i}\frac{\varphi}{2}} \end{pmatrix} \begin{pmatrix} \mathrm{e}^{\frac{\epsilon}{2}} & 0 \\ 0 & \mathrm{e}^{-\frac{\epsilon}{2}} \end{pmatrix} \begin{pmatrix} \mathrm{e}^{\mathbf{k}\frac{\varsigma}{2}} & 0 \\ 0 & \mathrm{e}^{-\mathbf{k}\frac{\varsigma}{2}} \end{pmatrix},$$
$$\begin{pmatrix} \mathrm{e}^{\mathbf{i}\frac{\psi^q}{2}} & 0 \\ 0 & \mathrm{e}^{-\mathbf{i}\frac{\psi^q}{2}} \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{\mathbf{i}\frac{\psi}{2}} & 0 \\ 0 & \mathrm{e}^{-\mathbf{i}\frac{\psi}{2}} \end{pmatrix} \begin{pmatrix} \mathrm{e}^{\frac{\varepsilon}{2}} & 0 \\ 0 & \mathrm{e}^{-\frac{\varepsilon}{2}} \end{pmatrix} \begin{pmatrix} \mathrm{e}^{\frac{\omega}{2}} & 0 \\ 0 & \mathrm{e}^{-\frac{\omega}{2}} \end{pmatrix} \begin{pmatrix} \mathrm{e}^{\mathbf{j}\frac{\chi}{2}} & 0 \\ 0 & \mathrm{e}^{-\mathbf{j}\frac{\chi}{2}} \end{pmatrix} \right\}.$$

Therefore, the Cartan decomposition $Sp(1,1) = K^q A^q K^q$ of the element $\mathfrak{q} \in Sp(1,1)$ is

$$\begin{aligned} g \equiv \mathfrak{q}(\varphi^q, \theta^q, \psi^q) &= \mathfrak{q}(\varphi, \epsilon, \varsigma, \theta, \tau, \phi, \psi, \varepsilon, \omega, \chi) \\ &= \begin{pmatrix} \mathrm{e}^{\mathbf{i}\frac{\varphi}{2}} & 0 \\ 0 & \mathrm{e}^{-\mathbf{i}\frac{\varphi}{2}} \end{pmatrix} \begin{pmatrix} \mathrm{e}^{\frac{\epsilon}{2}} & 0 \\ 0 & \mathrm{e}^{-\frac{\epsilon}{2}} \end{pmatrix} \begin{pmatrix} \mathrm{e}^{\mathbf{k}\frac{\varsigma}{2}} & 0 \\ 0 & \mathrm{e}^{-\mathbf{k}\frac{\varsigma}{2}} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} & \mathbf{i}\sin\frac{\theta}{2} \\ \mathbf{i}\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \\ &\quad \times \begin{pmatrix} \cosh\frac{\tau}{2} & \sinh\frac{\tau}{2} \\ \sinh\frac{\tau}{2} & \cosh\frac{\tau}{2} \end{pmatrix} \begin{pmatrix} \cos\frac{\phi}{2} & \mathbf{i}\sin\frac{\phi}{2} \\ \mathbf{i}\sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{pmatrix} \end{aligned}$$

$$\times \begin{pmatrix} e^{\mathbf{i}\frac{\psi}{2}} & 0 \\ 0 & e^{-\mathbf{i}\frac{\psi}{2}} \end{pmatrix} \begin{pmatrix} e^{\frac{\varepsilon}{2}} & 0 \\ 0 & e^{-\frac{\varepsilon}{2}} \end{pmatrix} \begin{pmatrix} e^{\frac{\omega}{2}} & 0 \\ 0 & e^{-\frac{\omega}{2}} \end{pmatrix} \begin{pmatrix} e^{\mathbf{j}\frac{\chi}{2}} & 0 \\ 0 & e^{-\mathbf{j}\frac{\chi}{2}} \end{pmatrix}$$

$$= \begin{pmatrix} e^{\mathbf{i}\frac{\varphi^q}{2}} & 0 \\ 0 & e^{-\mathbf{i}\frac{\varphi^q}{2}} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta^q}{2} & \mathbf{i}\sin\frac{\theta^q}{2} \\ \mathbf{i}\sin\frac{\theta^q}{2} & \cos\frac{\theta^q}{2} \end{pmatrix} \begin{pmatrix} e^{\mathbf{i}\frac{\psi^q}{2}} & 0 \\ 0 & e^{-\mathbf{i}\frac{\psi^q}{2}} \end{pmatrix}, \tag{18}$$

where

$$\left.\begin{aligned} \theta^q &= \theta + \phi - \mathbf{i}\tau, \\ \varphi^q &= \varphi - \mathbf{i}\epsilon + \mathbf{j}\varsigma, \\ \psi^q &= \psi - \mathbf{i}\varepsilon - \mathbf{i}\omega + \mathbf{k}\chi \end{aligned}\right\} \tag{19}$$

are *quaternion Euler angles*[3]. Hence it immediately follows that the element (18) is a quaternionic extension of (14).

2.2. *Differential operators on the group* $Sp(1,1)$

Let $\Omega(t)$ be the one-parameter subgroup of $Sp(1,1)$ and let $\omega(t)$ be a matrix from the group $\Omega(t)$. The operators of the right regular representation of $Sp(1,1)$, corresponding to the elements of the subgroup $\Omega(t)$, transfer quaternion functions $f(\mathfrak{q})$ into $R(\omega(t))f(\mathfrak{q}) = f(\mathfrak{q}\omega(t))$. For that reason, the infinitesimal operator of the right regular representation $R(\mathfrak{q})$, associated with one-parameter subgroup $\Omega(t)$, transfers the function $f(\mathfrak{q})$ into $\frac{\mathrm{d}f(\mathfrak{q}\omega(t))}{\mathrm{d}t}$ at $t = 0$.

Let us denote quaternion Euler angles of the element $\mathfrak{q}\omega(t)$ via $\varphi^q(t)$, $\theta^q(t)$, $\psi^q(t)$. Then there is an equality

$$\left.\frac{\mathrm{d}f(\mathfrak{q}\omega(t))}{\mathrm{d}t}\right|_{t=0} = \frac{\partial f}{\partial\varphi^q}(\varphi^q(0))' + \frac{\partial f}{\partial\theta^q}(\theta^q(0))' + \frac{\partial f}{\partial\psi^q}(\psi^q(0))'.$$

The infinitesimal operator J_ω, corresponding to the subgroup $\Omega(t)$, has a form

$$J_\omega = (\varphi^q(0))'\frac{\partial}{\partial\varphi^q} + (\theta^q(0))'\frac{\partial}{\partial\theta^q} + (\psi^q(0))'\frac{\partial}{\partial\psi^q}.$$

Let us calculate infinitesimal operators $J^q_{\omega_1}$, $J^q_{\omega_2}$, $J^q_{\omega_3}$ corresponding to the quaternion subgroups $\Omega^q_1, \Omega^q_2, \Omega^q_3$. The quaternion subgroups Ω^q_i $(i = 1,2,3)$ arise from the fact that all the ten parameters of $Sp(1,1)$ can be divided in three groups according the Cartan decomposition (18) for the element $\mathfrak{q} \in Sp(1,1)$. The subgroup Ω^q_3 consists of the matrices

$$\omega_3(t^q) = \begin{pmatrix} e^{\mathbf{i}\frac{t^q}{2}} & 0 \\ 0 & e^{-\mathbf{i}\frac{t^q}{2}} \end{pmatrix},$$

where the variable t^q has the form of quaternionic angles. Let $\mathfrak{q} = \mathfrak{q}(\varphi^q, \theta^q, \psi^q)$ be a matrix with quaternion Euler angles (the matrix (18)) $\varphi^q = \varphi - \mathbf{i}\epsilon + \mathbf{j}\varsigma$, $\theta^q = \theta + \phi - \mathbf{i}\tau$, $\psi^q = \psi - \mathbf{i}\varepsilon - \mathbf{i}\omega + \mathbf{k}\chi$. Therefore, Euler angles of the matrix $\mathfrak{q}\omega_3(t^q)$ equal $\varphi^q, \theta^q, \psi^q = t - \mathbf{i}t - \mathbf{i}t + \mathbf{k}t$. Hence, it follows that

$$\begin{aligned} &\varphi'(0) = 0, && \epsilon'(0) = 0, && \omega'(0) = -\mathbf{i}, && \theta'(0) = 0, \\ &\phi'(0) = 0, && \tau'(0) = 0, && \psi'(0) = 1, && \varepsilon'(0) = -\mathbf{i}, \\ &\varsigma'(0) = \mathbf{j}, && \chi'(0) = \mathbf{k}. \end{aligned}$$

[3] Quaternion Euler angles of $\mathbf{Spin}_+(1,4) \simeq Sp(1,1)$ contain complex Euler angles $\theta^c = \theta - \mathbf{i}\tau$, $\varphi^c = \varphi - \mathbf{i}\epsilon$, $\psi^c = \psi - \mathbf{i}\varepsilon$ of the group $\mathbf{Spin}_+(1,3) \simeq SL(2,\mathbb{C})$ as a particular case (for more details see [30]).

So, the operator $J^q_{\omega_3}$, corresponding to the subgroup Ω^c_3, has the form

$$J^q_{\omega_3} = \frac{\partial}{\partial\psi} - \mathbf{i}\frac{\partial}{\partial\varepsilon} - \mathbf{i}\frac{\partial}{\partial\omega} + \mathbf{k}\frac{\partial}{\partial\chi}. \tag{20}$$

Whence

$$M_3 = \frac{\partial}{\partial\psi}, \qquad N_3 = \frac{\partial}{\partial\varepsilon}, \qquad P_3 = \frac{\partial}{\partial\chi}, \qquad P_0 = \frac{\partial}{\partial\omega}. \tag{21}$$

Let us calculate the infinitesimal operator $J^q_{\omega_1}$ corresponding to the quaternion subgroup Ω^q_1. The subgroup Ω^q_1 consists of the matrices

$$\omega_1(t^q) = \begin{pmatrix} \cos\frac{t^q}{2} & \mathbf{i}\sin\frac{t^q}{2} \\ \mathbf{i}\sin\frac{t^q}{2} & \cos\frac{t^q}{2} \end{pmatrix}.$$

The Euler angles of these matrices equal to 0, $t^q = t + et - \mathbf{i}t$, 0, e is the double unit. Let us represent the matrix $\mathfrak{q}\omega_1(t^q)$ by the following product:

$$\mathfrak{q}\omega_1(t^q) = \begin{pmatrix} \cos\frac{\theta^q}{2}\,\mathrm{e}^{\mathbf{i}\frac{(\varphi^q+\psi^q)}{2}} & \mathbf{i}\sin\frac{\theta^q}{2}\,\mathrm{e}^{\mathbf{i}\frac{(\varphi^q-\psi^q)}{2}} \\ \mathbf{i}\sin\frac{\theta^q}{2}\,\mathrm{e}^{\mathbf{i}\frac{(\psi^q-\varphi^q)}{2}} & \cos\frac{\theta^q}{2}\,\mathrm{e}^{-\mathbf{i}\frac{(\varphi^q+\psi^q)}{2}} \end{pmatrix} \begin{pmatrix} \cos\frac{t^q}{2} & \mathbf{i}\sin\frac{t^q}{2} \\ \mathbf{i}\sin\frac{t^q}{2} & \cos\frac{t^q}{2} \end{pmatrix}.$$

Multiplying the matrices on the right-hand side of the latter expression, we obtain

$$\cos\theta^q(t) = \cos\theta^q\cos t^q - \sin\theta^q\sin t^q\cos\psi^q, \tag{22}$$

$$\mathrm{e}^{\mathbf{i}\varphi^q(t)} = \mathrm{e}^{\mathbf{i}\varphi^q}\frac{\sin\theta^q\cos t^q + \cos\theta^q\sin t^q\cos\psi^q + \mathbf{i}\sin t^q\sin\psi^q}{\sin\theta^q(t)}, \tag{23}$$

$$\mathrm{e}^{\mathbf{i}\frac{[\varphi^q(t)+\psi^q(t)]}{2}} = \mathrm{e}^{\mathbf{i}\frac{\varphi^q}{2}}\frac{\cos\frac{\theta^q}{2}\cos\frac{t^q}{2}\mathrm{e}^{\mathbf{i}\frac{\psi^q}{2}} - \sin\frac{\theta^q}{2}\sin\frac{t^q}{2}\mathrm{e}^{-\mathbf{i}\frac{\psi^q}{2}}}{\cos\frac{\theta^q(t)}{2}}. \tag{24}$$

For the calculation of derivatives $\varphi'(t)$, $\epsilon'(t)$, $\omega'(t)$, $\theta'(t)$, $\phi'(t)$, $\tau'(t)$, $\psi'(t)$, $\varepsilon'(t)$, $\varsigma'(t)$, $\chi'(t)$ at $t = 0$ we must differentiate on t both parts of each equality from (22) to (24). At this point, we have $\varphi(0) = \varphi$, $\epsilon(0) = \epsilon$, …, $\chi(0) = \chi$.

So, let us differentiate both parts of (22). As a result, we obtain

$$\begin{aligned} -\sin\theta^q(t)[\theta'(t) + e\phi'(t) - \mathbf{i}\tau'(t)] = {} & -\cos\theta^q\sin t^q(1 + e - \mathbf{i}) \\ & - \sin\theta^q\cos t^q\cos\psi^q(1 + e - \mathbf{i}). \end{aligned}$$

Taking $t = 0$, we find that

$$\theta'(0) + e\phi'(0) - \mathbf{i}\tau'(0) = \cos\psi^q(1 + e - \mathbf{i}).$$

Whence

$$\theta'(0) = \cos\psi^q, \qquad \phi'(0) = \cos\psi^q, \qquad \tau'(0) = \cos\psi^q.$$

Differentiating now both parts of (23) and taking $t = 0$, we obtain

$$\varphi'(0) - \mathbf{i}\epsilon'(0) + \mathbf{j}\varsigma'(0) = \frac{\sin\psi^q(1 + e - \mathbf{i})}{\sin\theta^q}.$$

Therefore,

$$\varphi'(0) = \frac{\sin\psi^q}{\sin\theta^q}, \qquad \epsilon'(0) = \frac{\sin\psi^q}{\sin\theta^q}, \qquad \varsigma'(0) = \frac{\sin\psi^q}{\sin\theta^q}.$$

Further, differentiating both parts of (24) and taking $t = 0$, we find that

$$\psi'(0) - \mathbf{i}\varepsilon'(0) - \mathbf{i}\omega'(0) + \mathbf{j}\chi'(0) = (-1 - e + \mathbf{i}) \cot\theta^q \sin\psi^q$$

and

$$\psi'(0) = \varepsilon'(0) = \chi'(0) = -\cot\theta^q \sin\psi^q, \qquad \omega'(0) = 0.$$

In such a way, we have

$$J^q_{\omega_1} = M_1 + P_1 - \mathbf{i}N_1, \tag{25}$$

where

$$M_1 = \cos\psi^q \frac{\partial}{\partial\theta} + \frac{\sin\psi^q}{\sin\theta^q}\frac{\partial}{\partial\varphi} - \cot\theta^q \sin\psi^q \frac{\partial}{\partial\psi}, \tag{26}$$

$$N_1 = \cos\psi^q \frac{\partial}{\partial\tau} + \frac{\sin\psi^q}{\sin\theta^q}\frac{\partial}{\partial\epsilon} - \cot\theta^q \sin\psi^q \frac{\partial}{\partial\varepsilon}, \tag{27}$$

$$P_1 = \cos\psi^q \frac{\partial}{\partial\phi} + \frac{\sin\psi^q}{\sin\theta^q}\frac{\partial}{\partial\varsigma} - \cot\theta^q \sin\psi^q \frac{\partial}{\partial\chi}. \tag{28}$$

Let us calculate now an infinitesimal operator $J^q_{\omega_2}$ corresponding to the quaternion subgroup Ω^q_2. The subgroup Ω^q_2 consists of the matrices

$$\omega_2(t^q) = \begin{pmatrix} \cos\frac{t^q}{2} & -\sin\frac{t^q}{2} \\ \sin\frac{t^q}{2} & \cos\frac{t^q}{2} \end{pmatrix},$$

where the Euler angles equal correspondingly to $0, t^c = t - \mathbf{i}t + \mathbf{j}t, 0$. It is obvious that the matrix $\mathfrak{q}\omega_2(t^q)$ can be represented by the product

$$\mathfrak{q}\omega_1(t^q) = \begin{pmatrix} \cos\frac{\theta^q}{2}\,\mathrm{e}^{\mathbf{i}\frac{(\varphi^q+\psi^q)}{2}} & \mathbf{i}\sin\frac{\theta^q}{2}\,\mathrm{e}^{\mathbf{i}\frac{(\varphi^q-\psi^q)}{2}} \\ \mathbf{i}\sin\frac{\theta^q}{2}\,\mathrm{e}^{\mathbf{i}\frac{(\psi^q-\varphi^q)}{2}} & \cos\frac{\theta^q}{2}\,\mathrm{e}^{-\mathbf{i}\frac{(\varphi^q+\psi^q)}{2}} \end{pmatrix} \begin{pmatrix} \cos\frac{t^q}{2} & -\sin\frac{t^q}{2} \\ \sin\frac{t^q}{2} & \cos\frac{t^q}{2} \end{pmatrix}.$$

Multiplying the matrices on the right-hand side of this equality, we see that Euler angles of the product $\mathfrak{q}\omega_2(t^q)$ are related by the formulae

$$\cos\theta^q(t) = \cos\theta^q \cos t^q + \sin\theta^q \sin t^q \sin\psi^q, \tag{29}$$

$$\mathrm{e}^{\mathbf{i}\varphi^q(t)} = \mathrm{e}^{\mathbf{i}\varphi^q} \frac{\sin\theta^q \cos t^q - \cos\theta^q \sin t^q \sin\psi^q + \mathbf{i}\sin t^q \cos\psi^q}{\sin\theta^q(t)}, \tag{30}$$

$$\mathrm{e}^{\mathbf{i}\frac{[\varphi^q(t)+\psi^q(t)]}{2}} = \mathrm{e}^{\mathbf{i}\frac{\varphi^q}{2}} \frac{\cos\frac{\theta^q}{2}\cos\frac{t^q}{2}\mathrm{e}^{\mathbf{i}\frac{\psi^q}{2}} + \sin\frac{\theta^q}{2}\sin\frac{t^q}{2}\mathrm{e}^{-\mathbf{i}\frac{\psi^q}{2}}}{\cos\frac{\theta^q(t)}{2}}. \tag{31}$$

Differentiating on t both parts of the each equalities (29)–(31) and taking $t = 0$, we obtain

$$\theta'(0) = \tau'(0) = \phi'(0) = -\sin\psi^q,$$
$$\varphi'(0) = \epsilon'(0) = \varsigma'(0) = \frac{\cos\psi^q}{\sin\theta^q},$$
$$\psi'(0) = \varepsilon'(0) = \chi'(0) = -\cot\theta^q \cos\psi^q, \qquad \omega'(0) = 0.$$

Therefore, for the subgroup Ω^q_2 we have

$$J^q_{\omega_2} = M_2 - \mathbf{i}N_2 + \mathbf{j}P_2, \tag{32}$$

where

$$M_2 = -\sin\psi^q \frac{\partial}{\partial\theta} + \frac{\cos\psi^q}{\cos\theta^q}\frac{\partial}{\partial\varphi} - \cot\theta^q \sin\psi^q \frac{\partial}{\partial\psi}, \tag{33}$$

$$N_2 = -\sin\psi^q \frac{\partial}{\partial\tau} + \frac{\cos\psi^q}{\sin\theta^q}\frac{\partial}{\partial\epsilon} - \cot\theta^q \cos\psi^q \frac{\partial}{\partial\varepsilon}, \tag{34}$$

$$P_2 = -\sin\psi^q \frac{\partial}{\partial\phi} + \frac{\cos\psi^q}{\sin\theta^q}\frac{\partial}{\partial\varsigma} - \cot\theta^q \cos\psi^q \frac{\partial}{\partial\chi}. \tag{35}$$

Let us introduce an auxiliary quaternion angle $\psi_1^q = \psi - \mathbf{i}\varepsilon + \mathbf{k}\chi$. It is easy to see that $\psi^q = \psi_1^q - \mathbf{i}\omega$; therefore, ψ_1^q is the part of ψ^q. Further, taking into account expressions (21), (26)–(28) and (33)–(35), we can rewrite the operators (20), (25), (32) in the form[4]

$$J^q_{\omega_1} = \cos\psi^q \frac{\partial}{\partial\theta^q} + \frac{\sin\psi^q}{\sin\theta^q}\frac{\partial}{\partial\varphi^q} - \cot\theta^q \sin\psi^q \frac{\partial}{\partial\psi_1^q}, \tag{36}$$

$$J^q_{\omega_2} = -\sin\psi^q \frac{\partial}{\partial\theta^q} + \frac{\cos\psi^q}{\sin\theta^q}\frac{\partial}{\partial\varphi^q} - \cot\theta^q \cos\psi^q \frac{\partial}{\partial\psi_1^q}, \tag{37}$$

$$J^q_{\omega_3} = \frac{\partial}{\partial\psi^q}, \tag{38}$$

$$j^q_{\omega_1} = \cos\dot\psi^q \frac{\partial}{\partial\dot\theta^q} + \frac{\sin\dot\psi^q}{\sin\dot\theta^q}\frac{\partial}{\partial\dot\varphi^q} - \cot\dot\theta^q \sin\dot\psi^q \frac{\partial}{\partial\dot\psi_1^q}, \tag{39}$$

$$j^q_{\omega_2} = -\sin\dot\psi^q \frac{\partial}{\partial\dot\theta^q} + \frac{\cos\dot\psi^q}{\sin\dot\theta^q}\frac{\partial}{\partial\dot\varphi^q} - \cot\dot\theta^q \cos\dot\psi^q \frac{\partial}{\partial\dot\psi_1^q}, \tag{40}$$

$$j^q_{\omega_3} = \frac{\partial}{\partial\dot\psi^q}, \tag{41}$$

where

$$\frac{\partial}{\partial\theta^q} = \frac{\partial}{\partial\theta} + \frac{\partial}{\partial\phi} + \mathbf{i}\frac{\partial}{\partial\tau}, \qquad \frac{\partial}{\partial\dot\theta^q} = \frac{\partial}{\partial\theta} - \frac{\partial}{\partial\phi} - \mathbf{i}\frac{\partial}{\partial\tau},$$

$$\frac{\partial}{\partial\varphi^q} = \frac{\partial}{\partial\varphi} + \mathbf{i}\frac{\partial}{\partial\epsilon} + \mathbf{j}\frac{\partial}{\partial\varsigma}, \qquad \frac{\partial}{\partial\dot\varphi^q} = \frac{\partial}{\partial\varphi} - \mathbf{i}\frac{\partial}{\partial\epsilon} - \mathbf{j}\frac{\partial}{\partial\varsigma},$$

$$\frac{\partial}{\partial\psi^q} = \frac{\partial}{\partial\psi} + \mathbf{i}\frac{\partial}{\partial\varepsilon} + \mathbf{i}\frac{\partial}{\partial\omega} + \mathbf{k}\frac{\partial}{\partial\chi}, \qquad \frac{\partial}{\partial\dot\psi^q} = \frac{\partial}{\partial\psi} - \mathbf{i}\frac{\partial}{\partial\varepsilon} - \mathbf{i}\frac{\partial}{\partial\omega} - \mathbf{k}\frac{\partial}{\partial\chi},$$

$$\frac{\partial}{\partial\psi_1^q} = \frac{\partial}{\partial\psi} + \mathbf{i}\frac{\partial}{\partial\varepsilon} + \mathbf{k}\frac{\partial}{\partial\chi}, \qquad \frac{\partial}{\partial\dot\psi_1^q} = \frac{\partial}{\partial\psi} - \mathbf{i}\frac{\partial}{\partial\varepsilon} - \mathbf{k}\frac{\partial}{\partial\chi}.$$

Using expressions (36)–(38), we see that for the first Casimir operator F of the group $SO_0(1,4)$ there exists the following equality:

$$-F = -P_0^2 - \boldsymbol{N}^2 + \boldsymbol{P}^2 + \boldsymbol{M}^2 = \left(J^q_{\omega_1}\right)^2 + \left(J^q_{\omega_2}\right)^2 + \left(J^q_{\omega_3}\right)^2,$$

or

$$-F = \frac{\partial^2}{\partial\theta^{q\,2}} + \cot\theta^q \frac{\partial}{\partial\theta^q} + \frac{1}{\sin^2\theta^q}\frac{\partial^2}{\partial\varphi^{q\,2}} - \frac{2\cos\theta^q}{\sin^2\theta^q}\frac{\partial^2}{\partial\varphi^q\partial\psi_1^q} + \cot^2\theta^q \frac{\partial^2}{\partial\psi_1^{q\,2}} + \frac{\partial^2}{\partial\psi^{q\,2}}. \tag{42}$$

[4] These operators look like as $SU(2)$ type (or $SU(2)\otimes SU(2)$ type) infinitesimal operators. However, it is easy to verify that they do not form a group, since $\psi^q \neq \psi_1^q$.

Matrix elements $t^{\sigma}_{mn}(\mathfrak{q}) = \mathfrak{M}^{\sigma}_{mn}(\varphi^q, \theta^q, \psi^q)$ of irreducible representations of the group $SO_0(1,4)$ are eigenfunctions of the operator (42):

$$[-F + \sigma(\sigma+3)]\mathfrak{M}^{\sigma}_{mn}(\mathfrak{q}) = 0, \tag{43}$$

where

$$\mathfrak{M}^{\sigma}_{mn}(\mathfrak{q}) = \exp\left(-\mathbf{i}\left(m\varphi^q + n\left(\psi^q_1 - \mathbf{i}\omega\right)\right)\right)\mathfrak{Z}^{\sigma}_{mn}(\cos\theta^q), \tag{44}$$

since $\psi^q = \psi^q_1 - \mathbf{i}\omega$. Here, $\mathfrak{M}^{\sigma}_{mn}(\mathfrak{q})$ are general matrix elements of the representations of $SO_0(1,4)$ and $\mathfrak{Z}^{\sigma}_{mn}(\cos\theta^q)$ are *hyperspherical functions*. Substituting the functions (44) into (43) and taking into account the operator (42), we arrive at the following differential equation:

$$\frac{\mathrm{d}^2\mathfrak{Z}^{\sigma}_{mn}(\cos\theta^q)}{\mathrm{d}\theta^{q\,2}} + \cot\theta^q\frac{\mathrm{d}\mathfrak{Z}^{\sigma}_{mn}(\cos\theta^q)}{\mathrm{d}\theta^q} - \frac{m^2}{\sin^2\theta^q}\mathfrak{Z}^{\sigma}_{mn}(\cos\theta^q) + \frac{2mn\cos\theta^q}{\sin^2\theta^q}\mathfrak{Z}^{\sigma}_{mn}(\cos\theta^q)$$
$$- n^2\cot^2\theta^q\mathfrak{Z}^{\sigma}_{mn}(\cos\theta^q) - n^2\mathfrak{Z}^{\sigma}_{mn}(\cos\theta^q) + \sigma(\sigma+3)\mathfrak{Z}^{\sigma}_{mn}(\cos\theta^q) = 0,$$

or

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}\theta^{q\,2}} + \cot\theta^q\frac{\mathrm{d}}{\mathrm{d}\theta^q} - \frac{m^2+n^2-2mn\cos\theta^q}{\sin^2\theta^q} + \sigma(\sigma+3)\right]\mathfrak{Z}^{\sigma}_{mn}(\cos\theta^q) = 0.$$

After substitution $z = \cos\theta^q$ this equation can be rewritten as

$$\left[(1-z^2)\frac{\mathrm{d}^2}{\mathrm{d}z^2} - 2z\frac{\mathrm{d}}{\mathrm{d}z} - \frac{m^2+n^2-2mnz}{1-z^2} + \sigma(\sigma+3)\right]\mathfrak{Z}^{\sigma}_{mn}(z) = 0. \tag{45}$$

The latter equation has three singular points $-1, +1, \infty$. It is a Fuchsian equation. Indeed, denoting $w(z) = \mathfrak{Z}^{\sigma}_{mn}(z)$, we write equation (45) in the form

$$\frac{\mathrm{d}^2w(z)}{\mathrm{d}z^2} - p(z)\frac{\mathrm{d}w(z)}{\mathrm{d}z} + q(z)w(z) = 0, \tag{46}$$

where

$$p(z) = \frac{2z}{(1-z)(1+z)}, \qquad q(z) = \frac{\sigma(\sigma+3)(1-z^2) - m^2 - n^2 + 2mnz}{(1-z)^2(1+z)^2}.$$

Let us find solutions of (46). Applying the substitution

$$t = \frac{1-z}{2}, \qquad w(z) = t^{\frac{|m-n|}{2}}(1-t)^{\frac{|m+n|}{2}}v(t),$$

we arrive at hypergeometric equation

$$t(1-t)\frac{\mathrm{d}^2v}{\mathrm{d}t^2} + [c - (a+b+1)t]\frac{\mathrm{d}v}{\mathrm{d}t} - abv(t) = 0, \tag{47}$$

where

$$a = \sigma + 3 + \tfrac{1}{2}(|m-n| + |m+n|),$$
$$b = -\sigma + \tfrac{1}{2}(|m-n| + |m+n|),$$
$$c = |m-n| + 1.$$

Therefore, a solution of (47) is

$$v(t) = C_1\,{}_2F_1\left(\begin{matrix}a, b\\ c\end{matrix}\middle|\, t\right) + C_2t^{1-c}\,{}_2F_1\left(\begin{matrix}b-c+1, a-c+1\\ 2-c\end{matrix}\middle|\, t\right).$$

Coming back to initial variable, we obtain

$$w(z) = C_1 \left(\frac{1-z}{2}\right)^{\frac{|m-n|}{2}} \left(\frac{1+z}{2}\right)^{\frac{|m+n|}{2}} \times {}_2F_1\left(\begin{matrix}\sigma+3+\frac{1}{2}(|m-n|+|m+n|), -\sigma+\frac{1}{2}(|m-n|+|m+n|) \\ |m-n|+1\end{matrix}\middle| \frac{1-z}{2}\right) + C_2 \left(\frac{1-z}{2}\right)^{-\frac{|m-n|}{2}} \left(\frac{1+z}{2}\right)^{\frac{|m+n|}{2}} \times {}_2F_1\left(\begin{matrix}-\sigma+\frac{1}{2}(|m+n|-|m-n|), \sigma+3+\frac{1}{2}(|m+n|-|m-n|) \\ 1-|m-n|\end{matrix}\middle| \frac{1-z}{2}\right). \tag{48}$$

Thus, from (48) it follows that the function $\mathfrak{Z}^{\sigma}_{mn}$ can be represented by the following particular solution:

$$\mathfrak{Z}^{\sigma}_{mn}(\cos\theta^q) = C_1 \sin^{|m-n|}\frac{\theta^q}{2}\cos^{|m+n|}\frac{\theta^q}{2} \times {}_2F_1\left(\begin{matrix}\sigma+3+\frac{1}{2}(|m-n|+|m+n|), -\sigma+\frac{1}{2}(|m-n|+|m+n|) \\ |m-n|+1\end{matrix}\middle| \sin^2\frac{\theta^q}{2}\right). \tag{49}$$

In section 4 and 5 we will give more explicit expressions for the functions $\mathfrak{Z}^{\sigma}_{mn}(\cos\theta^q)$ via the multiple hypergeometric series.

Finally, using formulae (39)–(41), we can obtain the same differential equation for the function $\mathfrak{Z}^{\sigma}_{\dot{m}\dot{n}}(\cos\dot{\theta}^q)$. All the calculations in this case are analogous to the previous calculations for $\mathfrak{Z}^{\sigma}_{mn}(\cos\theta^q)$.

2.3. Homogeneous spaces of $SO_0(1,4)$

Before introducing the spherical functions on the group $SO_0(1,4)$ it is useful to give a general definition for spherical functions on the group G. Let $T(g)$ be an irreducible representation of the group G in the space L and let H be a subgroup of G. The vector $\boldsymbol{\xi}$ in the space L is called *an invariant with respect to the subgroup* H if for all $h \in H$ the equality $T(h)\boldsymbol{\xi} = \boldsymbol{\xi}$ holds. The representation $T(g)$ is called *a representation of the class* 1 *with respect to the subgroup* H if in its space there are non-null vectors which are invariant with respect to H. At this point, a contraction of $T(g)$ onto its subgroup H is unitary:

$$(T(h)\boldsymbol{\xi}_1, T(h)\boldsymbol{\xi}_2) = (\boldsymbol{\xi}_1, \boldsymbol{\xi}_2).$$

Hence, it follows that a function

$$f(g) = (T(g)\boldsymbol{\eta}, \boldsymbol{\xi})$$

corresponds to each vector $\boldsymbol{\eta} \in L$. $f(g)$ are called *spherical functions of the representation* $T(g)$ with respect to H.

Spherical functions can be considered as functions on homogeneous spaces $\mathcal{M} = G/H$. In its turn, a homogeneous space $\mathcal{M}$ of the group G has the following properties:

(a) It is a topological space on which the group G acts continuously, that is, let y be a point in $\mathcal{M}$, then gy is defined and is again a point in $\mathcal{M}$ ($g \in G$).
(b) This action is transitive, that is, for any two points y_1 and y_2 in $\mathcal{M}$ it is always possible to find a group element $g \in G$ such that $y_2 = gy_1$.

There is a one-to-one correspondence between the homogeneous spaces of G and the coset spaces of G. Let H_0 be a maximal subgroup of G which leaves the point y_0 invariant, $hy_0 = y_0, h \in H_0$, then H_0 is called *the stabilizer of* y_0. Representing now any group element of G in the form $g = g_c h$, where $h \in H_0$ and $g_c \in G/H_0$, we see that, by virtue of the transitivity property, any point $y \in \mathcal{M}$ can be given by $y = g_c h y_0 = g_c y$. Hence, it follows that the elements g_c of the coset space give a parametrization of $\mathcal{M}$. The mapping $\mathcal{M} \leftrightarrow G/H_0$ is continuous since the group multiplication is continuous and the action on $\mathcal{M}$ is continuous by definition. The stabilizers H and H_0 of two different points y and y_0 are conjugate, since from $H_0 g_0 = g_0$, $y_0 = g^{-1}y$, it follows that $gH_0g^{-1}y = y$, that is, $H = gH_0g^{-1}$.

Coming back to the de Sitter group $G = SO_0(1,4)$, we see that there are the following homogeneous spaces of $SO_0(1,4)$ depending on the stabilizer H. First of all, when $H = 0$ the homogeneous space $\mathcal{M}_{10}$ coincides with *a group manifold* $\mathfrak{S}_{10}$ of $SO_0(1,4)$. Therefore, $\mathfrak{S}_{10}$ is a maximal homogeneous space of the de Sitter group. Further, when $H = \Omega^q_\psi$, where Ω^q_ψ is a group of diagonal matrices

$$\begin{pmatrix} \mathrm{e}^{\frac{\mathrm{i}\psi q}{2}} & 0 \\ 0 & \mathrm{e}^{-\frac{\mathrm{i}\psi q}{2}} \end{pmatrix},$$

the homogeneous space $\mathcal{M}_6$ coincides with a *two-dimensional quaternion sphere* S^q_2, $\mathcal{M}_6 = S^q_2 \sim Sp(1,1)\big/\Omega^q_\psi$.[5]

We obtain the following homogeneous space $\mathcal{M}_4$ when the stabilizer H coincides with a maximal compact subgroup $K = SO(4)$ of $SO_0(1,4)$. In this case, we arrive at the upper sheet of a four-dimensional hyperboloid $\mathcal{M}_4 = H^4 \sim SO_0(1,4)/SO(4)$. The upper sheet H^4_+ of the two-sheeted hyperboloid H^4 can be understood as a quotient space $SO_0(1,4)/SO(4)$. Indeed, let us consider the upper sheet H^4_+ of H^4:

$$H^4_+: \quad x_0^2 - x_1^2 - x_2^2 - x_3^2 - x_4^2 = 1, \qquad x_0 > 0 \tag{50}$$

and the point $x^0 = (1,0,0,0,0)$ on H^4_+. The group $SO_0(1,4)$ transfers the hyperboloid H^4_+ into itself. Besides, for any two points x' and x'' of H^4_+ there is such an element $g \in SO_0(1,4)$ that $gx' = x''$, that is, $SO_0(1,4)$ is a transitive transformation group of the homogeneous space. The set of elements from $SO_0(1,4)$, leaving the point x^0 invariant, coincides with the subgroup $SO(4)$. Therefore, H^4_+ is homeomorphic to the quotient space $SO_0(1,4)/SO(4)$. It should be noted that *a four-dimensional Lobatchevski space* $\mathcal{L}^4$, called also *a de Sitter space*, is realized on the hyperboloid H^4_+.[6]

In the case $x_0^2 - x_1^2 - x_2^2 - x_3^2 - x^4 = 0$, we arrive at a cone C^4 which can also be considered as a homogeneous space of $SO_0(1,4)$. Usually, only the upper sheets H^4_+ and C^4_+ are considered in applications.

The following homogeneous space $\mathcal{M}_3$ of $SO_0(1,4)$ is a three-dimensional real sphere $S^3 \sim SO(4)/SO(3)$. In contrast to the previous homogeneous spaces, the sphere S^3 coincides with a quotient space $SO_0(1,4)/P$, where P is a minimal parabolic subgroup of $SO_0(1,4)$. From the Iwasawa decompositions $SO_0(1,4) = KNA$ and $P = MNA$, where $M = SO(3)$, N and A are nilpotent and commutative subgroups of $SO_0(1,4)$, it follows that $SO_0(1,4)/P = KNA/MNA \sim K/M \sim SO(4)/SO(3)$.

[5] When the stabilizer H is a compact group, the homogeneous space $\mathcal{M} = G/H$ is called *a Riemannian symmetric space* [16]. When H is a non-compact group, we arrive at the non-Riemannian spaces. The homogeneous space $\mathcal{M}_6 = S^q_2 \sim Sp(1,1)/\Omega^q_\psi$ is the non-Riemannian space, since the stabilizer $H = \Omega^q_\psi$ is non-compact subgroup of $Sp(1,1)$. Quaternion and anti-quaternion spheres were studied by Rozenfel'd [21].

[6] It is obvious that among all the homogeneous spaces of $SO_0(1,4)$ the space H^4_+ is the most important for physics. In accordance with modern cosmology, H^4_+ is understood as a spacetime endowed with a global topology of constant negative curvature (the de Sitter universe).

A minimal homogeneous space $\mathcal{M}_2$ of $SO_0(1,4)$ is a two-dimensional real sphere $S^2 \sim SO(3)/SO(2)$.

Taking into account the list of homogeneous spaces of $SO_0(1,4)$, we now introduce the following types of spherical functions $f(\mathfrak{q})$ on the de Sitter group:

- $f(\mathfrak{q}) = \mathfrak{M}^{\sigma}_{mn}(\mathfrak{q}) = \mathrm{e}^{-\mathbf{i}m\varphi^q}\mathfrak{Z}^{\sigma}_{mn}(\cos\theta^q)\,\mathrm{e}^{-\mathbf{i}n\psi^q}$. This function is defined on the group manifold $\mathfrak{S}_{10}$ of $SO_0(1,4)$. It is the most general spherical function on the group $SO_0(1,4)$. In this case, $f(\mathfrak{q})$ depends on all the ten parameters of $SO_0(1,4)$ and for that reason it should be called as *a function on the de Sitter group*. An explicit form of $\mathfrak{M}^{\sigma}_{mn}(\mathfrak{q})$ (respectively $\mathfrak{M}^{\dot\sigma}_{\dot m\dot n}(\mathfrak{q})$) for finite-dimensional representations and of $\mathfrak{M}^{-\frac{3}{2}+\mathbf{i}\rho,l_0}_{mn}(\mathfrak{q})$ (resp. $\mathfrak{M}^{-\frac{3}{2}-\mathbf{i}\rho,l_0}_{\dot m\dot n}(\mathfrak{q})$) for infinite-dimensional representations of $SO_0(1,4)$ will be given in sections 4 and 5, respectively.
- $f(\varphi^q,\theta^q) = \mathfrak{M}^{m}_{\sigma}(\varphi^q,\theta^q,0) = \mathrm{e}^{-\mathbf{i}\varphi^q}\mathfrak{Z}^{m}_{\sigma}(\cos\theta^q)$. This function is defined on the homogeneous space $\mathcal{M}_6 = S^2_q \sim Sp(1,1)/\Omega^q_{\psi}$, that is, on the surface of the two-dimensional quaternion sphere S^q_2. The function $\mathfrak{M}^{m}_{\sigma}(\varphi^q,\theta^q,0)$ is a five-dimensional analogue of the usual spherical function $Y^m_l(\varphi,\theta)$ defined on the surface of the real 2-sphere S_2. In its turn, the function $f(\dot\varphi^q,\dot\theta^q) = \mathfrak{M}^{\dot m}_{\dot\sigma}(\dot\varphi^q,\dot\theta^q,0)$ is defined on the surface of the dual quaternion sphere $\dot S^2_q$. An explicit form of the functions $\mathfrak{M}^{m}_{\sigma}(\varphi^q,\theta^q,0)$ $\big(\mathfrak{M}^{\dot m}_{\dot\sigma}(\dot\varphi^q,\dot\theta^q,0)\big)$ and $\mathfrak{M}^{m}_{-\frac{3}{2}+\mathbf{i}\rho,l_0}(\varphi^q,\theta^q,0)$ $\big(\mathfrak{M}^{\dot m}_{-\frac{3}{2}-\mathbf{i}\rho,l_0}(\dot\varphi^q,\dot\theta^q,0)\big)$ will be given in section 4 and 5.
- $f(\epsilon,\tau,\varepsilon,\omega) = \mathfrak{M}^{\sigma}_{mn}(\epsilon,\tau,\varepsilon,\omega) = \mathrm{e}^{\mathbf{i}m\epsilon}\mathfrak{P}^{\sigma}_{mn}(\cosh\tau)\,\mathrm{e}^{\mathbf{i}n(\varepsilon+\omega)}$. This function is defined on the homogeneous space $\mathcal{M}_4 = H^4_+ \sim SO_0(1,4)/SO(4)$, that is, on the upper sheet of the hyperboloid $x_0^2-x_1^2-x_2^2-x_3^2-x_4^2 = 1$. An explicit form of the functions $\mathfrak{M}^{\sigma}_{mn}(\epsilon,\tau,\varepsilon,\omega)$ $\big(\mathfrak{M}^{\dot\jmath}_{\dot m\dot n}(\epsilon,\tau,\varepsilon,\omega)\big)$ and $\mathfrak{M}^{-\frac{3}{2}+\mathbf{i}\rho}_{mn}(\epsilon,\tau,\varepsilon,\omega)$ $\big(\mathfrak{M}^{-\frac{3}{2}-\mathbf{i}\rho}_{\dot m\dot n}(\epsilon,\tau,\varepsilon,\omega)\big)$ will be given in section 4 and 5.
- $f(\varphi,\theta,\psi) = \mathfrak{M}^{\sigma}_{mn}(\varphi,\theta,\psi) = \mathrm{e}^{-\mathbf{i}m\varphi}P^{\sigma}_{mn}(\cos\theta)\,\mathrm{e}^{-\mathbf{i}n\psi}$ (or $f(\varsigma,\phi,\chi) = \mathfrak{M}^{\sigma}_{mn}(\varsigma,\phi,\chi) = \mathrm{e}^{-\mathbf{i}m\varsigma}P^{\sigma}_{mn}(\cos\phi)\,\mathrm{e}^{-\mathbf{i}n\chi}$). This function is defined on the homogeneous space $\mathcal{M}_3 \sim S^3 = SO(4)/SO(3)$, that is, on the surface of the real 3-sphere $x_0^2+x_1^2+x_2^2+x_3^2 = 1$. In essence, we come here to representations of $SO_0(1,4)$ restricted to the subgroup $SO(4)$.
- $f(\varphi,\theta) = \mathfrak{M}^{m}_{l}(\varphi,\theta,0) = \mathrm{e}^{-\mathbf{i}m\varphi}P^{m}_{\sigma}(\cos\theta) \sim Y^m_{\sigma}(\varphi,\theta)$ (or $f(\varsigma,\phi) = \mathfrak{M}^{m}_{\sigma}(\varsigma,\phi,0) = \mathrm{e}^{-\mathbf{i}m\varsigma}P^{m}_{\sigma}(\cos\phi) \sim Y^m_{\sigma}(\varsigma,\phi)$). This function is defined on the homogeneous space $\mathcal{M}_2 = S^2 \sim SO(3)/SO(2)$, that is, on the surface of the real 2-sphere S^2. We come here to the most degenerate representations of $SO_0(1,4)$ restricted to the subgroup $SO(3)$.

3. Spherical functions on the group $SO(4)$

As is known, the group $SO(4)$ is a maximal compact subgroup of $SO_0(1,4)$. $SO(4)$ corresponds to basis elements $\boldsymbol{M} = (M_1, M_2, M_3)$ and $\boldsymbol{P} = (P_1, P_2, P_3)$ of the algebra $\mathfrak{so}(1,4)$:

$$[M_k, M_l] = \mathbf{i}\varepsilon_{klm}M_m, \qquad [M_k, P_l] = \mathbf{i}\varepsilon_{klm}P_m, \qquad [P_k, P_l] = \mathbf{i}\varepsilon_{klm}M_m. \tag{51}$$

Introducing linear combinations $\boldsymbol{V} = (\boldsymbol{M}+\boldsymbol{P})/2$ and $\boldsymbol{V}' = (\boldsymbol{M}-\boldsymbol{P})/2$, we obtain

$$[V_k, V_l] = \mathbf{i}\varepsilon_{klm}V_m, \qquad [V'_k, V'_l] = \mathbf{i}\varepsilon_{klm}V'_m. \tag{52}$$

The operators $\boldsymbol{V}$ and $\boldsymbol{V}'$ form bases of the two independent algebras $\mathfrak{so}(3)$. It means that $SO(4)$ is isomorphic to a direct product $SO(3)\otimes SO(3)$.

A universal covering of $SO(4)$ is $\mathbf{Spin}(4) \simeq SU(2) \otimes SU(2)$. The one-parameter subgroups of $\mathbf{Spin}(4)$ are

$$m_{12}(\psi) = \begin{pmatrix} \mathrm{e}^{\mathbf{i}\frac{\psi}{2}} & 0 \\ 0 & \mathrm{e}^{-\mathbf{i}\frac{\psi}{2}} \end{pmatrix}, \qquad m_{13}(\varphi) = \begin{pmatrix} \mathrm{e}^{\mathbf{i}\frac{\varphi}{2}} & 0 \\ 0 & \mathrm{e}^{-\mathbf{i}\frac{\varphi}{2}} \end{pmatrix}, \qquad m_{23}(\theta) = \begin{pmatrix} \cos\frac{\theta}{2} & \mathbf{i}\sin\frac{\theta}{2} \\ \mathbf{i}\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix},$$

$$p_{14}(\chi) = \begin{pmatrix} \mathrm{e}^{\mathbf{i}\frac{\chi}{2}} & 0 \\ 0 & \mathrm{e}^{-\mathbf{i}\frac{\chi}{2}} \end{pmatrix}, \qquad p_{24}(\varsigma) = \begin{pmatrix} \mathrm{e}^{\mathbf{i}\frac{\varsigma}{2}} & 0 \\ 0 & \mathrm{e}^{-\mathbf{i}\frac{\varsigma}{2}} \end{pmatrix}, \qquad p_{34}(\phi) = \begin{pmatrix} \cos\frac{\phi}{2} & \mathbf{i}\sin\frac{\phi}{2} \\ \mathbf{i}\sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{pmatrix},$$

where

$$\begin{aligned} &0 \leqslant \theta \leqslant \pi, && 0 \leqslant \phi \leqslant \pi, \\ &0 \leqslant \varphi < 2\pi, && 0 \leqslant \varsigma < 2\pi, \\ &-2\pi \leqslant \psi < 2\pi, && -2\pi \leqslant \chi < 2\pi, \end{aligned}$$

A fundamental representation of the group $\mathbf{Spin}(4) \simeq SU(2) \otimes SU(2)$ is defined by the matrix (16).

On the group $SO(4)$ there exist the following Laplace–Beltrami operators:

$$\boldsymbol{V}^2 = V_1^2 + V_2^2 + V_3^2 = \tfrac{1}{4}(\boldsymbol{M}^2 + \boldsymbol{P}^2 + 2\boldsymbol{MP}), \tag{53}$$

$$\boldsymbol{V}'^2 = V_1'^2 + V_2'^2 + V_3'^2 = \tfrac{1}{4}(\boldsymbol{M}^2 + \boldsymbol{P}^2 - 2\boldsymbol{MP}). \tag{54}$$

At this point, we see that operators (53), (54) contain Casimir operators $\boldsymbol{M}^2 + \boldsymbol{P}^2$, $\boldsymbol{MP}$ of the group $SO(4)$. Using expressions (17), we obtain a Euler parametrization of the Laplace–Beltrami operators,

$$\begin{aligned} \boldsymbol{V}^2 &= \frac{\partial^2}{\partial\theta^{e2}} + \cot\theta^e\frac{\partial}{\partial\theta^e} + \frac{1}{\sin^2\theta^e}\left[\frac{\partial^2}{\partial\varphi^{e2}} - 2\cos\theta^e\frac{\partial}{\partial\varphi^e}\frac{\partial}{\partial\psi^e} + \frac{\partial^2}{\partial\psi^{e2}}\right], \\ \boldsymbol{V}'^2 &= \frac{\partial^2}{\partial\dot\theta^{e2}} + \cot\dot\theta^e\frac{\partial}{\partial\dot\theta^e} + \frac{1}{\sin^2\dot\theta^e}\left[\frac{\partial^2}{\partial\dot\varphi^{e2}} - 2\cos\dot\theta^e\frac{\partial}{\partial\dot\varphi^e}\frac{\partial}{\partial\dot\psi^e} + \frac{\partial^2}{\partial\dot\psi^{e2}}\right]. \end{aligned} \tag{55}$$

Here, $\dot\theta^e = \theta - \phi$, $\dot\varphi^e = \varphi - \varsigma$, $\dot\psi^e = \psi - \chi$ are conjugate double angles.

Matrix elements $t^l_{mn}(g) = \mathfrak{M}^l_{mn}(\varphi^e, \theta^e, \psi^e)$ of irreducible representations of the group $SO(4)$ are eigenfunctions of the operators (55),

$$\begin{aligned} &[\boldsymbol{V}^2 + l(l+1)]\mathfrak{M}^l_{mn}(\varphi^e, \theta^e, \psi^e) = 0, \\ &[\boldsymbol{V}'^2 + \dot l(\dot l + 1)]\mathfrak{M}^{\dot l}_{\dot m\dot n}(\dot\varphi^e, \dot\theta^e, \dot\psi^e) = 0, \end{aligned} \tag{56}$$

where

$$\begin{aligned} \mathfrak{M}^l_{mn}(g) &= \mathrm{e}^{-\mathbf{i}(m\varphi^e + n\psi^e)}\mathfrak{Z}^l_{mn}(\cos\theta^e), \\ \mathfrak{M}^{\dot l}_{\dot m\dot n}(g) &= \mathrm{e}^{\mathbf{i}(\dot m\dot\varphi^e + \dot n\dot\psi^e)}\mathfrak{Z}^{\dot l}_{\dot m\dot n}(\cos\dot\theta^e). \end{aligned} \tag{57}$$

Here, $\mathfrak{M}^l_{mn}(g)$ are general matrix elements of the representations of $SO(4)$ and $\mathfrak{Z}^l_{mn}(\cos\theta^e)$ are *hyperspherical functions* of $SO(4)$. Substituting the functions (57) into (56) and taking into account the operators (55) and substitutions $z = \cos\theta^e$, $\overset{*}{z} = \cos\dot\theta^e$, we come to the following differential equations:

$$\left[(1-z^2)\frac{\mathrm{d}^2}{\mathrm{d}z^2} - 2z\frac{\mathrm{d}}{\mathrm{d}z} - \frac{m^2 + n^2 - 2mnz}{1-z^2} + l(l+1)\right]\mathfrak{Z}^l_{mn}(z) = 0, \tag{58}$$

$$\left[(1-\overset{*}{z}{}^2)\frac{\mathrm{d}^2}{\mathrm{d}\overset{*}{z}{}^2} - 2\overset{*}{z}\frac{\mathrm{d}}{\mathrm{d}\overset{*}{z}} - \frac{\dot m^2 + \dot n^2 - 2\dot m\dot n\overset{*}{z}}{1-\overset{*}{z}{}^2} + \dot l(\dot l+1)\right]\mathfrak{Z}^{\dot l}_{\dot m\dot n}(\overset{*}{z}) = 0. \tag{59}$$

The latter equations have three singular points $-1, +1, \infty$. Equations (58), (59) are Fuchsian equations. Indeed, denoting $w(z) = \mathfrak{Z}^l_{mn}(z)$, we write equation (58) in the form

$$\frac{\mathrm{d}^2 w(z)}{\mathrm{d}z^2} - p(z)\frac{\mathrm{d}w(z)}{\mathrm{d}z} + q(z)w(z) = 0, \tag{60}$$

where

$$p(z) = \frac{2z}{(1-z)(1+z)}, \qquad q(z) = \frac{l(l+1)(1-z^2) - m^2 - n^2 + 2mnz}{(1-z)^2(1+z)^2}.$$

The solution of (60) is

$$\begin{aligned} w(z) = C_1 &\left(\frac{1-z}{2}\right)^{\frac{|m-n|}{2}} \left(\frac{1+z}{2}\right)^{\frac{|m+n|}{2}} \\ &\times {}_2F_1\left(\begin{matrix} l+1+\frac{1}{2}(|m-n|+|m+n|), -l+\frac{1}{2}(|m-n|+|m+n|) \\ |m-n|+1 \end{matrix}\,\middle|\, \frac{1-z}{2}\right) \\ &+ C_2 \left(\frac{1-z}{2}\right)^{-\frac{|m-n|}{2}} \left(\frac{1+z}{2}\right)^{\frac{|m+n|}{2}} \\ &\times {}_2F_1\left(\begin{matrix} -l+\frac{1}{2}(|m+n|-|m-n|), l+1+\frac{1}{2}(|m+n|-|m-n|) \\ 1-|m-n| \end{matrix}\,\middle|\, \frac{1-z}{2}\right). \end{aligned} \tag{61}$$

It is obvious that a solution of (59) has the analogous structure.

Let us now consider spherical functions $f(g)$ and homogeneous spaces $\mathcal{M} = SO(4)/H$ of the group $SO(4)$ depending on the stabilizer H. First of all, when $H = 0$ the homogeneous space $\mathcal{M}_6$ coincides with *a group manifold* $\mathfrak{K}_6$ of $SO(4)$. Therefore, $\mathfrak{K}_6$ is a maximal homogeneous space of the group $SO(4)$. Further, when $H = \Omega^e_\psi$, where Ω^e_ψ is a group of diagonal matrices

$$\begin{pmatrix} \mathrm{e}^{\frac{\mathrm{i}\psi^e}{2}} & 0 \\ 0 & \mathrm{e}^{-\frac{\mathrm{i}\psi^e}{2}} \end{pmatrix},$$

the homogeneous space $\mathcal{M}_4$ coincides with a *two-dimensional double sphere* S^e_2, $\mathcal{M}_4 = S^e_2 \sim \mathbf{Spin}(4)/\Omega^e_\psi$. The sphere S^e_2 can be constructed from the quantities $z_k = x_k + e y_k$, $\overset{*}{z}_k = x_k - e y_k$ $(k = 1, 2, 3)$ as follows:

$$S^e_2\colon \quad z_1^2 + z_2^2 + z_3^2 = \mathbf{x}^2 + \mathbf{y}^2 + 2e\mathbf{xy} = r^2, \tag{62}$$

where e is *a double unit*, $e^2 = 1$. The conjugate (dual) sphere $\dot{S}^e_2$ is

$$\dot{S}^e_2\colon \quad \overset{*}{z}_1{}^2 + \overset{*}{z}_2{}^2 + \overset{*}{z}_3{}^2 = \mathbf{x}^2 + \mathbf{y}^2 - 2e\mathbf{xy} = \overset{*}{r}{}^2. \tag{63}$$

We obtain the following homogeneous space $\mathcal{M}_3$ when the stabilizer H coincides with a subgroup $SO(3)$. In this case, we have a three-dimensional sphere $\mathcal{M}_3 = S^3 \sim SO(4)/SO(3)$ in the space $\mathbb{R}^4$.

Finally, a minimal homogeneous space $\mathcal{M}_2$ of $SO(4)$ is a two-dimensional real sphere $S_2 \sim SO(3)/SO(2)$. All the homogeneous spaces of $SO(4)$ are symmetric Riemannian spaces.

Taking into account the list of homogeneous spaces of $SO(4)$, we now introduce the following types of spherical functions $f(g)$ on the group $SO(4)$:

- $f(g) = \mathfrak{M}^l_{mn}(g)$. This function is defined on the group manifold $\mathfrak{K}_6$ of $SO(4)$. It is the most general spherical function on the group $SO(4)$. In this case, $f(g)$ depends on all the six parameters of $SO(4)$ and for that reason it should be called as *a function on the group* $SO(4)$.
- $f(\varphi^e, \theta^e) = \mathfrak{M}^m_l(\varphi^e, \theta^e, 0)$. This function is defined on the homogeneous space $\mathcal{M}_4 = S^e_2 \sim SO(4)\big/\Omega^e_\psi$, that is, on the surface of the two-dimensional double sphere S^e_2. The function $\mathfrak{M}^m_l(\varphi^e, \theta^e, 0)$ is a four-dimensional analogue of the usual spherical function $Y^m_l(\varphi, \theta)$ defined on the surface of the real 2-sphere S^2. In its turn, the function $f(\dot\varphi^e, \dot\theta^e) = \mathfrak{M}^{\dot m}_{\dot l}(\dot\varphi^e, \dot\theta^e, 0)$ is defined on the surface of the dual sphere $\dot S^e_2$.
- $f(\varphi, \theta, \psi) = \mathrm{e}^{-\mathbf{i}m\varphi} P^l_{mn}(\cos\theta)\,\mathrm{e}^{-\mathbf{i}n\psi}$ (or $f(\varsigma, \phi, \chi) = \mathrm{e}^{-\mathbf{i}m\varsigma} P^l_{mn}(\cos\phi)\,\mathrm{e}^{-\mathbf{i}n\chi}$). This function is defined on the homogeneous space $\mathcal{M}_3 \sim S^3 = SO(4)/SO(3)$, that is, on the surface of the real 3-sphere $x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1$.
- $f(\varphi, \theta) = \mathrm{e}^{-\mathbf{i}m\varphi} P^m_l(\cos\theta) \sim Y^m_l(\varphi, \theta)$ (or $f(\varsigma, \phi) = \mathrm{e}^{-\mathbf{i}m\varsigma} P^m_l(\cos\phi) \sim Y^m_l(\varsigma, \phi)$). This function is defined on the homogeneous space $\mathcal{M}_2 = S^2 \sim SO(3)/SO(2)$, that is, on the surface of the real 2-sphere S^2. We come here to the most degenerate representations of $SO(4)$ restricted to the subgroup $SU(2)$.

First, let us consider spherical functions $f(g) = \mathfrak{M}^l_{mn}(g) = \mathrm{e}^{-\mathbf{i}m\varphi^e} \mathfrak{Z}^l_{mn}(\cos\theta^e)\,\mathrm{e}^{-\mathbf{i}n\psi^e}$ on the group manifold $\mathfrak{K}_6$ of $SO(4)$. The Laplace–Beltrami operators $\triangle_L(\mathfrak{K}_6)$ and $\overline{\triangle}_L(\mathfrak{K}_6)$ coincide with (53) and (54). Spherical functions of the first type $f(g) = \mathfrak{M}^l_{mn}(g)$ ($f(\dot g) = \mathfrak{M}^{\dot l}_{\dot m\dot n}(\dot g)$) are eigenfunctions of the operator $\triangle_L(\mathfrak{K}_6)$ ($\overline{\triangle}_L(\mathfrak{K}_6)$). With the aim to find an explicit form of hyperspherical functions on $\mathfrak{Z}^l_{mn}(\cos\theta^e)$, we will use an addition theorem for generalized spherical functions $P^l_{mn}(\cos\theta)$ of the group $SU(2)$ [31]:

$$\mathrm{e}^{-\mathbf{i}(m\varphi+n\psi)} P^l_{mn}(\cos\theta) = \sum_{k=-l}^{l} \mathrm{e}^{-\mathbf{i}k\varphi_2} P^l_{mk}(\cos\theta_1) P^l_{kn}(\cos\theta_2), \tag{64}$$

where the angles $\varphi, \psi, \theta, \theta_1, \varphi_2, \theta_2$ are related by the formulae

$$\cos\theta = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\cos\varphi_2, \tag{65}$$

$$\mathrm{e}^{\mathbf{i}\varphi} = \frac{\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2\cos\varphi_2 + \mathbf{i}\sin\theta_2\sin\varphi_2}{\sin\theta}, \tag{66}$$

$$\mathrm{e}^{\frac{\mathbf{i}(\varphi+\psi)}{2}} = \frac{\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\,\mathrm{e}^{\mathbf{i}\frac{\varphi_2}{2}} - \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\,\mathrm{e}^{-\mathbf{i}\frac{\varphi_2}{2}}}{\cos\frac{\theta}{2}}. \tag{67}$$

Let $\cos(\theta + \phi) = \cos\theta^e$ and $\varphi_2 = 0$, then formulae (65)–(67) take the form

$$\cos\theta^e = \cos\theta\cos\phi - \sin\theta\sin\phi,$$
$$\mathrm{e}^{\mathbf{i}\varphi} = \frac{\sin\theta\cos\phi + \cos\theta\sin\phi}{\sin\theta^e} = 1,$$
$$\mathrm{e}^{\frac{\mathbf{i}(\varphi+\psi)}{2}} = \frac{\cos\frac{\theta}{2}\cos\frac{\phi}{2} - \sin\frac{\theta}{2}\sin\frac{\phi}{2}}{\cos\frac{\theta^e}{2}} = 1.$$

Hence, it follows that $\varphi = \psi = 0$ and formula (64) can be written as

$$\mathfrak{Z}^l_{mn}(\cos\theta^e) = \sum_{k=-l}^{l} P^l_{mk}(\cos\theta) P^l_{kn}(\cos\phi). \tag{68}$$

$\mathfrak{Z}^l_{mn}(\cos\theta^e)$ are *hyperspherical functions of the group* $SO(4)$.[7] Using an explicit expression for the function P^l_{mn} [30, 31], we obtain

$$\begin{aligned}\mathfrak{Z}^l_{mn}(\cos\theta^e) = {} & \sum_{k=-l}^{l} \mathbf{i}^{m+n-2k}\sqrt{\Gamma(l-m+1)\Gamma(l+m+1)\Gamma(l-k+1)\Gamma(l+k+1)} \\ & \times \cos^{2l}\frac{\theta}{2}\tan^{m-k}\frac{\theta}{2} \\ & \times \sum_{j=\max(0,k-m)}^{\min(l-m,l+k)} \frac{\mathbf{i}^{2j}\tan^{2j}\frac{\theta}{2}}{\Gamma(j+1)\Gamma(l-m-j+1)\Gamma(l+k-j+1)\Gamma(m-k+j+1)} \\ & \times \sqrt{\Gamma(l-n+1)\Gamma(l+n+1)\Gamma(l-k+1)\Gamma(l+k+1)}\cos^{2l}\frac{\phi}{2}\tan^{n-k}\frac{\phi}{2} \\ & \times \sum_{s=\max(0,k-n)}^{\min(l-n,l+k)} \frac{\mathbf{i}^{2s}\tan^{2s}\frac{\phi}{2}}{\Gamma(s+1)\Gamma(l-n-s+1)\Gamma(l+k-s+1)\Gamma(n-k+s+1)}.\end{aligned} \tag{69}$$

On the other hand, the function $\mathfrak{Z}^l_{mn}(\cos\theta^e)$ can be expressed via the hypergeometric function. Using hypergeometric-type formulae for P^l_{mn} [30, 31], we have at $m \geqslant n$

$$\begin{aligned}\mathfrak{Z}^l_{mn}(\cos\theta^e) = {} & \mathbf{i}^{m-n}\sqrt{\frac{\Gamma(l+m+1)\Gamma(l-n+1)}{\Gamma(l-m+1)\Gamma(l+n+1)}}\cos^{2l}\frac{\theta}{2}\cos^{2l}\frac{\phi}{2}\sum_{k=-l}^{l}\tan^{m-k}\frac{\theta}{2}\tan^{k-n}\frac{\phi}{2} \\ & \times {}_2F_1\left(\begin{matrix}m-l,-k-l\\ m-k+1\end{matrix}\middle|-\tan^2\frac{\theta}{2}\right){}_2F_1\left(\begin{matrix}k-l,-n-l\\ k-n+1\end{matrix}\middle|-\tan^2\frac{\phi}{2}\right), \\ & m\geqslant k,\quad k\geqslant n;\end{aligned} \tag{70}$$

$$\begin{aligned}\mathfrak{Z}^l_{mn}(\cos\theta^e) = {} & \sqrt{\frac{\Gamma(l+m+1)\Gamma(l-n+1)}{\Gamma(l-m+1)\Gamma(l+n+1)}}\cos^{2l}\frac{\theta}{2}\cos^{2l}\frac{\phi}{2}\sum_{k=-l}^{l}\mathbf{i}^{m+n-2k}\tan^{m-k}\frac{\theta}{2}\tan^{n-k}\frac{\phi}{2} \\ & \times {}_2F_1\left(\begin{matrix}m-l,-k-l\\ m-k+1\end{matrix}\middle|-\tan^2\frac{\theta}{2}\right){}_2F_1\left(\begin{matrix}n-l,-k-l\\ n-k+1\end{matrix}\middle|-\tan^2\frac{\phi}{2}\right), \\ & m\geqslant k,\quad n\geqslant k;\end{aligned} \tag{71}$$

and at $n \geqslant m$

$$\begin{aligned}\mathfrak{Z}^l_{mn}(\cos\theta^e) = {} & \mathbf{i}^{n-m}\sqrt{\frac{\Gamma(l-m+1)\Gamma(l+n+1)}{\Gamma(l+m+1)\Gamma(l-n+1)}}\cos^{2l}\frac{\theta}{2}\cos^{2l}\frac{\phi}{2}\sum_{k=-l}^{l}\tan^{k-m}\frac{\theta}{2}\tan^{n-k}\frac{\phi}{2} \\ & \times {}_2F_1\left(\begin{matrix}k-l,-m-l\\ k-m+1\end{matrix}\middle|-\tan^2\frac{\theta}{2}\right){}_2F_1\left(\begin{matrix}n-l,-k-l\\ n-k+1\end{matrix}\middle|-\tan^2\frac{\phi}{2}\right), \\ & k\geqslant m,\quad n\geqslant k;\end{aligned} \tag{72}$$

[7] The functions $\mathfrak{Z}^l_{mn}(\cos\theta^e)$ and $\mathfrak{Z}^{\dot{l}}_{\dot{m}\dot{n}}(\cos\dot{\theta}^e)$ form a representation of the type $(l,0)\oplus(0,\dot{l})$, that is, when $l=\dot{l}$. In the case of tensor representations, when $l\neq\dot{l}$, we arrive at the functions $\mathfrak{Z}^{l\dot{l}}_{mn;\dot{m}\dot{n}}(\cos\theta^e,\cos\dot{\theta}^e) = \mathfrak{Z}^l_{mn}(\cos\theta^e)\mathfrak{Z}^{\dot{l}}_{\dot{m}\dot{n}}(\cos\dot{\theta}^e)$ (*generalized hyperspherical functions of* $SO(4)$), which can be expressed via the product of the two generalized hypergeometric functions ${}_3F_2\left(\begin{matrix}\alpha,\beta,\gamma\\ \delta,\epsilon\end{matrix}\middle|x\right)$. In the case of Lorentz group, general solutions of relativistic wave equations for arbitrary spin chains (tensor representations) are defined via an expansion in generalized hyperspherical functions $\mathfrak{Z}^{l\dot{l}}_{mn;\dot{m}\dot{n}}(\cos\theta^c,\cos\dot{\theta}^c)$ of $SO_0(1,3)$, where $\theta^c,\dot{\theta}^c$ are complex Euler angles of $\mathbf{Spin}_+(1,3)\simeq SL(2,\mathbb{C})$ [28].

$$\mathfrak{Z}^l_{mn}(\cos\theta^e) = \sqrt{\frac{\Gamma(l-m+1)\Gamma(l+n+1)}{\Gamma(l+m+1)\Gamma(l-n+1)}}\cos^{2l}\frac{\theta}{2}\cos^{2l}\frac{\phi}{2}\sum_{k=-l}^{l}\mathbf{i}^{2k-m-n}\tan^{k-m}\frac{\theta}{2}\tan^{k-n}\frac{\phi}{2}$$
$$\times\,{}_2F_1\left(\begin{matrix}k-l,-m-l\\k-m+1\end{matrix}\middle|-\tan^2\frac{\theta}{2}\right){}_2F_1\left(\begin{matrix}k-l,-n-l\\k-n+1\end{matrix}\middle|-\tan^2\frac{\phi}{2}\right),$$
$$k\geqslant m,\quad k\geqslant n. \tag{73}$$

By way of example let us calculate matrix elements $\mathfrak{M}^l_{mn}(g) = \mathrm{e}^{-\mathbf{i}m\varphi^e}\mathfrak{Z}^l_{mn}(\cos\theta^e)\,\mathrm{e}^{-\mathbf{i}n\psi^e}$ at $l = 0, 1/2, 1$, where $\mathfrak{Z}^l_{mn}(\cos\theta^e)$ is defined via (69) or (70)–(73). The representation matrices at $l = 0, \frac{1}{2}, 1$ have the following form:

$$T_0(\varphi^e,\theta^e,\psi^e) = 1, \tag{74}$$

$$T_{\frac{1}{2}}(\varphi^e,\theta^e,\psi^e) = \begin{pmatrix}\mathfrak{M}^{\frac12}_{-\frac12-\frac12} & \mathfrak{M}^{\frac12}_{\frac12-\frac12}\\ \mathfrak{M}^{\frac12}_{-\frac12\frac12} & \mathfrak{M}^{\frac12}_{\frac12\frac12}\end{pmatrix} = \begin{pmatrix}\mathrm{e}^{\frac{\mathbf{i}}{2}\varphi^e}\mathfrak{Z}^{\frac12}_{-\frac12-\frac12}\mathrm{e}^{\frac{\mathbf{i}}{2}\psi^e} & \mathrm{e}^{\frac{\mathbf{i}}{2}\varphi^e}\mathfrak{Z}^{\frac12}_{-\frac12\frac12}\mathrm{e}^{-\frac{\mathbf{i}}{2}\psi^e}\\ \mathrm{e}^{-\frac{\mathbf{i}}{2}\varphi^e}\mathfrak{Z}^{\frac12}_{\frac12-\frac12}\mathrm{e}^{\frac{\mathbf{i}}{2}\psi^e} & \mathrm{e}^{-\frac{\mathbf{i}}{2}\varphi^e}\mathfrak{Z}^{\frac12}_{\frac12\frac12}\mathrm{e}^{-\frac{\mathbf{i}}{2}\psi^e}\end{pmatrix}$$
$$= \begin{pmatrix}\mathrm{e}^{\frac{\mathbf{i}}{2}\varphi^e}\cos\frac{\theta^e}{2}\,\mathrm{e}^{\frac{\mathbf{i}}{2}\psi^e} & \mathbf{i}\mathrm{e}^{\frac{\mathbf{i}}{2}\varphi^e}\sin\frac{\theta^e}{2}\,\mathrm{e}^{-\frac{\mathbf{i}}{2}\psi^e}\\ \mathbf{i}\,\mathrm{e}^{-\frac{\mathbf{i}}{2}\varphi^e}\sin\frac{\theta^e}{2}\,\mathrm{e}^{\frac{\mathbf{i}}{2}\psi^e} & \mathrm{e}^{-\frac{\mathbf{i}}{2}\varphi^e}\cos\frac{\theta^e}{2}\,\mathrm{e}^{-\frac{\mathbf{i}}{2}\psi^e}\end{pmatrix}$$
$$= \begin{pmatrix}\left[\cos\frac{\theta}{2}\cos\frac{\phi}{2}-\sin\frac{\theta}{2}\sin\frac{\phi}{2}\right]\mathrm{e}^{\frac{\mathbf{i}(\varphi+\varsigma+\psi+\chi))}{2}} & \mathbf{i}\left[\cos\frac{\theta}{2}\sin\frac{\phi}{2}+\sin\frac{\theta}{2}\cos\frac{\phi}{2}\right]\mathrm{e}^{\frac{\mathbf{i}(\varphi+\varsigma-\psi-\chi)}{2}}\\ \mathbf{i}\left[\cos\frac{\theta}{2}\sin\frac{\phi}{2}+\sin\frac{\theta}{2}\cos\frac{\phi}{2}\right]\mathrm{e}^{\frac{\mathbf{i}(-\varphi-\varsigma+\psi+\chi)}{2}} & \left[\cos\frac{\theta}{2}\cos\frac{\phi}{2}-\sin\frac{\theta}{2}\sin\frac{\phi}{2}\right]\mathrm{e}^{\frac{-\mathbf{i}(\varphi+\varsigma+\psi+\chi)}{2}}\end{pmatrix}, \tag{75}$$

$$T_1(\varphi^e,\theta^e,\psi^e) = \begin{pmatrix}\mathfrak{M}^1_{-1-1} & \mathfrak{M}^1_{-10} & \mathfrak{M}^1_{-11}\\ \mathfrak{M}^1_{0-1} & \mathfrak{M}^1_{00} & \mathfrak{M}^1_{01}\\ \mathfrak{M}^1_{1-1} & \mathfrak{M}^1_{10} & \mathfrak{M}^1_{11}\end{pmatrix} = \begin{pmatrix}\mathrm{e}^{\mathbf{i}\varphi^e}\mathfrak{Z}^1_{-1-1}\mathrm{e}^{\mathbf{i}\psi^e} & \mathrm{e}^{\mathbf{i}\varphi^e}\mathfrak{Z}^1_{-10} & \mathrm{e}^{\mathbf{i}\varphi^e}\mathfrak{Z}^1_{-11}\mathrm{e}^{-\mathbf{i}\psi^e}\\ \mathfrak{Z}^1_{0-1}\mathrm{e}^{\mathbf{i}\psi^e} & \mathfrak{Z}^1_{00} & \mathfrak{Z}^1_{01}\mathrm{e}^{-\mathbf{i}\psi^e}\\ \mathrm{e}^{-\mathbf{i}\varphi^e}\mathfrak{Z}^1_{1-1}\mathrm{e}^{\mathbf{i}\psi^e} & \mathrm{e}^{-\mathbf{i}\psi^e}\mathfrak{Z}^1_{10} & \mathrm{e}^{-\mathbf{i}\varphi^e}\mathfrak{Z}^1_{11}\mathrm{e}^{-\mathbf{i}\psi^e}\end{pmatrix}$$
$$= \begin{pmatrix}\mathrm{e}^{\mathbf{i}\varphi^e}\cos^2\frac{\theta^e}{2}\,\mathrm{e}^{\mathbf{i}\psi^e} & \frac{\mathbf{i}}{\sqrt2}\mathrm{e}^{\mathbf{i}\varphi^e}\sin\theta^e & -\mathrm{e}^{\mathbf{i}\varphi^e}\sin^2\frac{\theta^e}{2}\,\mathrm{e}^{-\mathbf{i}\psi^e}\\ \frac{\mathbf{i}}{\sqrt2}\sin\theta^e\,\mathrm{e}^{\mathbf{i}\psi^e} & \cos\theta^e & \frac{\mathbf{i}}{\sqrt2}\sin\theta^e\,\mathrm{e}^{-\mathbf{i}\psi^e}\\ -\mathrm{e}^{-\mathbf{i}\varphi^e}\sin^2\frac{\theta^e}{2}\,\mathrm{e}^{\mathbf{i}\psi^e} & \frac{\mathbf{i}}{\sqrt2}\mathrm{e}^{-\mathbf{i}\varphi^e}\sin\theta^e & \mathrm{e}^{-\mathbf{i}\varphi^e}\cos^2\frac{\theta^e}{2}\,\mathrm{e}^{-\mathbf{i}\psi^e}\end{pmatrix}$$
$$= \begin{pmatrix}\left[\cos^2\frac{\theta}{2}\cos^2\frac{\phi}{2}-\frac{\sin\theta\sin\phi}{2}+\sin^2\frac{\theta}{2}\sin^2\frac{\phi}{2}\right]\mathrm{e}^{\mathbf{i}(\varphi+\varsigma+\psi+\chi)} & \left[\frac{\mathbf{i}}{\sqrt2}(\cos\theta\sin\phi+\sin\theta\cos\phi)\right]\mathrm{e}^{\mathbf{i}(\varphi+\varsigma)}\\ \left[\frac{\mathbf{i}}{\sqrt2}(\cos\theta\sin\phi+\sin\theta\cos\phi)\right]\mathrm{e}^{\mathbf{i}(\psi+\chi)} & \cos\theta\cos\phi-\sin\theta\sin\phi\\ -\left[\cos^2\frac{\theta}{2}\sin^2\frac{\phi}{2}+\frac{\sin\theta\sin\phi}{2}+\sin^2\frac{\theta}{2}\cos^2\frac{\phi}{2}\right]\mathrm{e}^{\mathbf{i}(-\varphi-\varsigma+\psi+\chi)} & \left[\frac{\mathbf{i}}{\sqrt2}(\cos\theta\sin\phi+\sin\theta\cos\phi)\right]\mathrm{e}^{-\mathbf{i}(\varphi+\varsigma)}\\ -\left[\cos^2\frac{\theta}{2}\sin^2\frac{\phi}{2}+\frac{\sin\theta\sin\phi}{2}+\sin^2\frac{\theta}{2}\cos^2\frac{\phi}{2}\right]\mathrm{e}^{\mathbf{i}(\varphi+\varsigma-\psi-\chi)}\\ \left[\frac{\mathbf{i}}{\sqrt2}(\cos\theta\sin\phi+\sin\theta\cos\phi)\right]\mathrm{e}^{-\mathbf{i}(\psi+\chi)}\\ \left[\cos^2\frac{\theta}{2}\cos^2\frac{\phi}{2}-\frac{\sin\theta\sin\phi}{2}+\sin^2\frac{\theta}{2}\sin^2\frac{\phi}{2}\right]\mathrm{e}^{-\mathbf{i}(\varphi+\varsigma+\psi+\chi)}\end{pmatrix}. \tag{76}$$

Spherical functions of the second type $f(\varphi^e,\theta^e) = \mathfrak{M}^m_l(\varphi^e,\theta^e,0) = \mathrm{e}^{-\mathbf{i}m\varphi^e}\mathfrak{Z}^m_l(\cos\theta^e)$, where

$$\mathfrak{Z}^m_l(\cos\theta^e) = \sum_{k=-l}^{l}P^l_{mk}(\cos\theta)P^k_l(\cos\phi)$$

is *an associated hyperspherical function*, are defined on the surface of the double 2-sphere (62). The function $\mathfrak{Z}_l^m(\cos\theta^e)$ is an eigenfunction of the Laplace–Beltrami operator $\triangle_L\big(S_2^e\big)$ defined on the double 2-sphere,

$$\triangle_L\big(S_2^e\big) = \frac{\partial^2}{\partial\theta^{e2}} + \cot\theta^e\frac{\partial}{\partial\theta^e} + \frac{1}{\sin^2\theta^e}\frac{\partial^2}{\partial\varphi^{e2}}.$$

Hypergeometric-type formulae for $\mathfrak{Z}_l^m(\cos\theta^e)$ are

$$\mathfrak{Z}_l^m(\cos\theta^e) = \mathbf{i}^m\sqrt{\frac{\Gamma(l+m+1)}{\Gamma(l-m+1)}}\cos^{2l}\frac{\theta}{2}\cos^{2l}\frac{\phi}{2}\sum_{k=-l}^{l}\tan^{m-k}\frac{\theta}{2}\tan^k\frac{\phi}{2}$$
$$\times\,{}_2F_1\left(\begin{matrix}m-l,-k-l\\ m-k+1\end{matrix}\,\middle|\,-\tan^2\frac{\theta}{2}\right){}_2F_1\left(\begin{matrix}k-l,-l\\ k+1\end{matrix}\,\middle|\,-\tan^2\frac{\phi}{2}\right),\qquad m\geqslant k;$$

$$\mathfrak{Z}_l^m(\cos\theta^e) = \sqrt{\frac{\Gamma(l-m+1)}{\Gamma(l+m+1)}}\cos^{2l}\frac{\theta}{2}\cos^{2l}\frac{\phi}{2}\sum_{k=-l}^{l}\mathbf{i}^{2k-m}\tan^{k-m}\frac{\theta}{2}\tan^k\frac{\phi}{2}$$
$$\times\,{}_2F_1\left(\begin{matrix}k-l,-m-l\\ k-m+1\end{matrix}\,\middle|\,-\tan^2\frac{\theta}{2}\right){}_2F_1\left(\begin{matrix}k-l,-l\\ k+1\end{matrix}\,\middle|\,-\tan^2\frac{\phi}{2}\right),\qquad k\geqslant m.$$

We obtain an important particular case from the previous formulae at $m=n=0$. The function $Z_l(\cos\theta^e)\equiv\mathfrak{Z}_{00}^l(\cos\theta^e)$ is called *a zonal hyperspherical function*. The hypergeometric-type formula for $\mathfrak{Z}_l(\cos\theta^e)$ is

$$\mathfrak{Z}_l(\cos\theta^e) = \cos^{2l}\frac{\theta}{2}\cos^{2l}\frac{\phi}{2}\sum_{k=-l}^{l}\mathbf{i}^{2k}\tan^k\frac{\theta}{2}\tan^k\frac{\phi}{2}$$
$$\times\,{}_2F_1\left(\begin{matrix}k-l,-l\\ k+1\end{matrix}\,\middle|\,-\tan^2\frac{\theta}{2}\right){}_2F_1\left(\begin{matrix}k-l,-l\\ k+1\end{matrix}\,\middle|\,-\tan^2\frac{\phi}{2}\right).$$

In its turn, the function $f(\dot\varphi^e,\dot\theta^e) = \mathrm{e}^{\mathbf{i}n\dot\varphi^e}\mathfrak{Z}_l^n(\cos\dot\theta^e)$ (or $f(\dot\theta^e) = \mathfrak{Z}_l(\cos\dot\theta^e)$) is defined on the surface of dual sphere (63). Explicit expressions and hypergeometric-type formulae for $f(\dot\varphi^e,\dot\theta^e)$ are analogous to the previous expressions for $f(\varphi^e,\theta^e)$.

Spherical functions of the third type $f(\varphi,\theta,\psi) = \mathrm{e}^{-\mathbf{i}m\varphi}P_{mn}^l(\cos\theta)\,\mathrm{e}^{-\mathbf{i}n\psi}$ (or $f(\varsigma,\phi,\chi) = \mathrm{e}^{-\mathbf{i}m\varsigma}P_{mn}^l(\cos\phi)\,\mathrm{e}^{-\mathbf{i}n\chi}$) are defined on the surface of the real 3-sphere $S^3 = SO(4)/SO(3)$. These functions are general matrix elements of representations of the group $SO(3)$. Therefore, we have here representations of $SO(4)$ restricted to the subgroup $SO(3)$. Namely,

$$\hat T^l\downarrow_{SO(3)}^{SO(4)} = \sum_{m=0}^{l}\oplus Q^m, \tag{77}$$

where spherical functions $f(\varphi,\theta,\psi)$ of the representations Q^m of $SO(3)$ form an orthogonal basis in the Hilbert space $L^2(S^3)$. Various expressions and hypergeometric-type formulae for $f(\varphi,\theta,\psi)$ are given in [30, 31].

Finally, spherical functions of the fourth type $f(\varphi,\theta) = \mathrm{e}^{-\mathbf{i}m\varphi}P_l^m(\cos\theta)\sim Y_l^m(\varphi,\theta)$ (or $f(\varsigma,\phi) = \mathrm{e}^{-\mathbf{i}m\varsigma}P_l^m(\cos\phi)\sim Y_l^m(\varsigma,\phi)$) are defined on the surface of the real 2-sphere. We have here representations $\hat T^l\downarrow_{SO(3)}^{SO(4)}$ of the type (77), where associated spherical functions $f(\varphi,\theta)\sim Y_l^m(\varphi,\theta)$ of Q^m form an orthogonal basis in $L^2(S^3)$. These representations are the most degenerate for the group $SO(4)$.

4. Spherical functions of finite-dimensional representations of $SO_0(1,4)$

Let us come back to the de Sitter group $SO_0(1,4)$. It has been shown in section 1 that spherical functions of the first type $f(\mathfrak{q}) = \mathfrak{M}^{\sigma}_{mn}(\mathfrak{q}) = \mathrm{e}^{-\mathbf{i}m\varphi^q}\,\mathfrak{Z}^{\sigma}_{mn}(\cos\theta^q)\,\mathrm{e}^{-\mathbf{i}n\psi^q}$ are defined on the group manifold $\mathfrak{S}_{10}$ of $SO_0(1,4)$. With the aim to find an explicit form of hyperspherical function $\mathfrak{Z}^{\sigma}_{mn}(\cos\theta^q)$ of the group $SO_0(1,4)$, we will use the addition theorem defined by formulae (64)–(67). Let $\cos(\theta+\phi-\mathbf{i}\tau) = \cos(\theta^e-\mathbf{i}\tau) = \cos\theta^q$ and $\varphi_2 = 0$, then formulae (65)–(67) take the form

$$\cos\theta^q = \cos\theta^e\cosh\tau + \mathbf{i}\sin\theta^e\sinh\tau,$$
$$\mathrm{e}^{\mathbf{i}\varphi} = \frac{\sin\theta^e\cosh\tau - \mathbf{i}\cos\theta^e\sinh\tau}{\sin\theta^q} = 1,$$
$$\mathrm{e}^{\frac{\mathbf{i}(\varphi+\psi)}{2}} = \frac{\cos\frac{\theta^e}{2}\cosh\frac{\tau}{2} + \mathbf{i}\sin\frac{\theta^e}{2}\sinh\frac{\tau}{2}}{\cos\frac{\theta^q}{2}} = 1.$$

Hence, it follows that $\varphi = \psi = 0$ and formula (64) can be written as

$$\mathfrak{Z}^{\sigma}_{mn}(\cos\theta^q) = \sum_{k=-\sigma}^{\sigma}\mathfrak{Z}^{\sigma}_{mk}(\cos\theta^e)\mathfrak{P}^{\sigma}_{kn}(\cosh\tau), \tag{78}$$

where $\mathfrak{Z}^{\sigma}_{mn}(\cos\theta^e)$ is the hyperspherical function of the compact subgroup $SO(4)$ (see formula (68)):

$$\mathfrak{Z}^{\sigma}_{mk}(\cos\theta^e) = \sum_{t=-\sigma}^{\sigma}P^{\sigma}_{mt}(\cos\theta)\,P^{\sigma}_{tk}(\cos\phi).$$

It is easy to verify that if we take $\cos(\theta+\phi-\mathbf{i}\tau) = \cos(\phi+\theta^c) = \cos\theta^q$ and $\varphi_2 = 0$ in formulae (65)–(67), then we arrive at the function

$$\mathfrak{Z}^{\sigma}_{mn}(\cos\theta^q) = \sum_{k=-\sigma}^{\sigma}P^{\sigma}_{mk}(\cos\phi)\mathfrak{Z}^{\sigma}_{kn}(\cos\theta^c),$$

where

$$\mathfrak{Z}^{\sigma}_{kn}(\cos\theta^c) = \sum_{t=-\sigma}^{\sigma}P^{\sigma}_{kt}(\cos\theta)\mathfrak{P}^{\sigma}_{tn}(\cosh\tau)$$

is the hyperspherical function of the subgroup $SO_0(1,3)$. In such a way, the hyperspherical function $\mathfrak{Z}^{\sigma}_{mn}(\cos\theta^q)$ can be factorized with respect to the subgroups $SO(4)$ and $SO_0(1,3)$.

Further, taking into account the expression for $\mathfrak{Z}^{\sigma}_{mk}(\cos\theta^e)$, we can rewrite (78) in the following form:

$$\mathfrak{Z}^{\sigma}_{mn}(\cos\theta^q) = \sum_{k=-\sigma}^{\sigma}\sum_{t=-\sigma}^{\sigma}P^{\sigma}_{mt}(\cos\theta)\,P^{\sigma}_{tk}(\cos\phi)\mathfrak{P}^{\sigma}_{kn}(\cosh\tau). \tag{79}$$

Analogously, for the factorization of $\mathfrak{Z}^{\sigma}_{mn}(\cos\theta^q)$ with respect to the Lorentz subgroup $SO_0(1,3)$ we have

$$\mathfrak{Z}^{\sigma}_{mn}(\cos\theta^q) = \sum_{k=-\sigma}^{\sigma}\sum_{t=-\sigma}^{\sigma}P^{\sigma}_{mk}(\cos\phi)\,P^{\sigma}_{kt}(\cos\theta)\mathfrak{P}^{\sigma}_{tn}(\cosh\tau).$$

We consider here only the factorization of $\mathfrak{Z}^{\sigma}_{mn}(\cos\theta^q)$ with respect to the maximal compact subgroup $SO(4)$. Thus, formulae (78) and (79) define *a hyperspherical function of the*

de Sitter group $SO_0(1,4)$ with respect to $SO(4)$. Further, using (69), we obtain an explicit expression for $\mathfrak{Z}^{\sigma}_{mn}(\cos\theta^q)$,

$$\begin{aligned}\mathfrak{Z}^{\sigma}_{mn}(\cos\theta^q) = {} & \sum_{k=-\sigma}^{\sigma}\sum_{t=-\sigma}^{\sigma}\mathbf{i}^{m+k-2t}\sqrt{\Gamma(\sigma-m+1)\Gamma(\sigma+m+1)\Gamma(\sigma-t+1)\Gamma(\sigma+t+1)}\\ & \times\cos^{2\sigma}\frac{\theta}{2}\tan^{m-t}\frac{\theta}{2}\\ & \times\sum_{j=\max(0,t-m)}^{\min(\sigma-m,l+t)}\frac{\mathbf{i}^{2j}\tan^{2j}\frac{\theta}{2}}{\Gamma(j+1)\Gamma(\sigma-m-j+1)\Gamma(\sigma+t-j+1)\Gamma(m-t+j+1)}\\ & \times\sqrt{\Gamma(\sigma-k+1)\Gamma(\sigma+k+1)\Gamma(\sigma-t+1)\Gamma(\sigma+t+1)}\cos^{2\sigma}\frac{\phi}{2}\tan^{k-t}\frac{\phi}{2}\\ & \times\sum_{s=\max(0,t-k)}^{\min(\sigma-k,\sigma+t)}\frac{\mathbf{i}^{2s}\tan^{2s}\frac{\phi}{2}}{\Gamma(s+1)\Gamma(\sigma-k-s+1)\Gamma(\sigma+t-s+1)\Gamma(k-t+s+1)}\\ & \times\sqrt{\Gamma(\sigma-n+1)\Gamma(\sigma+n+1)\Gamma(\sigma-k+1)\Gamma(\sigma+k+1)}\cosh^{2\sigma}\frac{\tau}{2}\tanh^{n-k}\frac{\tau}{2}\\ & \times\sum_{p=\max(0,k-n)}^{\min(\sigma-n,\sigma+k)}\frac{\tanh^{2p}\frac{\tau}{2}}{\Gamma(p+1)\Gamma(\sigma-n-p+1)\Gamma(\sigma+k-p+1)\Gamma(n-k+p+1)}.\end{aligned}\tag{80}$$

It is obvious that the functions $\mathfrak{Z}^{\sigma}_{mn}(\cos\theta^q)$ can also be reduced to hypergeometric functions. Namely, these functions are expressed via the following multiple hypergeometric series[8]:

$$\begin{aligned}\mathfrak{Z}^{\sigma}_{mn}(\cos\theta^q) = {} & \sqrt{\frac{\Gamma(\sigma+m+1)\Gamma(\sigma-n+1)}{\Gamma(\sigma-m+1)\Gamma(\sigma+n+1)}}\cos^{2\sigma}\frac{\theta}{2}\cos^{2\sigma}\frac{\phi}{2}\cosh^{2\sigma}\frac{\tau}{2}\\ & \times\sum_{k=-\sigma}^{\sigma}\sum_{t=-\sigma}^{\sigma}\mathbf{i}^{m-k}\tan^{m-t}\frac{\theta}{2}\tan^{t-k}\frac{\phi}{2}\tanh^{k-n}\frac{\tau}{2}\\ & \times{}_2F_1\left(\begin{matrix}m-\sigma,-t-\sigma\\ m-t+1\end{matrix}\,\middle|\,-\tan^2\frac{\theta}{2}\right){}_2F_1\left(\begin{matrix}t-\sigma,-k-\sigma\\ t-k+1\end{matrix}\,\middle|\,-\tan^2\frac{\phi}{2}\right)\\ & \times{}_2F_1\left(\begin{matrix}k-\sigma,-n-\sigma\\ k-n+1\end{matrix}\,\middle|\,\tanh^2\frac{\tau}{2}\right),\\ & m\geqslant t,\quad t\geqslant k,\quad k\geqslant n;\end{aligned}\tag{81}$$

[8] The hyperspherical functions $\mathfrak{Z}^{\sigma}_{mn}(\cos\theta^q)$ of $SO_0(1,4)$, $\mathfrak{Z}^{l}_{mn}(\cos\theta^e)$ of $SO(4)$ and $\mathfrak{Z}^{l}_{mn}(\cos\theta^c)$ of $SO_0(1,3)$ can be written in the form of hypergeometric functions of many variables [3, 11]. So, the functions $\mathfrak{Z}^{l}_{mn}(\cos\theta^e)$ and $\mathfrak{Z}^{l}_{mn}(\cos\theta^c)$ can be expressed via the Appell functions, $\mathfrak{Z}^{l}_{mn}(\cos\theta^e)\sim F_4\big[\begin{smallmatrix}a_1,a_2\\ a_3,a_4\end{smallmatrix}\big|\,x_1;x_2\big]$ and $\mathfrak{Z}^{l}_{mn}(\cos\theta^c)\sim F_4\big[\begin{smallmatrix}a_1,a_2\\ a_3,a_4\end{smallmatrix}\big|\,x_1;y_1\big]$, where $x_1=\tan^2\theta/2$, $x_2=\tan^2\phi/2$, $y_1=\tanh^2\tau/2$. In its turn, the function $\mathfrak{Z}^{\sigma}_{mn}(\cos\theta^q)$ is reduced to the Lauricella function, $\mathfrak{Z}^{\sigma}_{mn}(\cos\theta^q)\sim\Psi_3\big[\begin{smallmatrix}a_1,a_2,a_3\\ a_4,a_5\end{smallmatrix}\big|\,x_1;x_2;y_1\big]$. From the relations $\mathbf{Spin}(4)\in C\ell^{+}_{4,0}\simeq C\ell_{0,3}$, where $C\ell_{0,3}$ is the algebra of double biquaternions with a double quaternionic division ring $\mathbb{K}\simeq\mathbb{H}\oplus\mathbb{H}$; $\mathbf{Spin}_+(1,3)\in C\ell^{+}_{1,3}\simeq C\ell_{3,0}$, where $C\ell_{3,0}$ is the algebra of complex biquaternions with a complex division ring $\mathbb{K}\simeq\mathbb{C}$; $\mathbf{Spin}_+(1,4)\in C\ell^{+}_{1,4}\simeq C\ell_{1,3}$, where $C\ell_{1,3}$ is the spacetime algebra with a quaternionic division ring $\mathbb{K}\simeq\mathbb{H}$, we see that there is a close relationship between hypercomplex angles of the group $\mathbf{Spin}_+(p,q)$, division rings of $C\ell^{+}_{p,q}$ on the one hand and hypergeometric functions of many variables on the other hand. A detailed consideration of this relationship comes beyond the framework of this paper and will be given in a separate work.

$$\mathfrak{Z}^{\sigma}_{mn}(\cos\theta^q) = \sqrt{\frac{\Gamma(\sigma+m+1)\Gamma(\sigma-n+1)}{\Gamma(\sigma-m+1)\Gamma(\sigma+n+1)}}\cos^{2\sigma}\frac{\theta}{2}\cos^{2\sigma}\frac{\phi}{2}\cosh^{2\sigma}\frac{\tau}{2}$$
$$\times\sum_{k=-\sigma}^{\sigma}\sum_{t=-\sigma}^{\sigma}\mathbf{i}^{m+k-2t}\tan^{m-t}\frac{\theta}{2}\tan^{k-t}\frac{\phi}{2}\tanh^{k-n}\frac{\tau}{2}$$
$$\times{}_2F_1\left(\begin{matrix}m-\sigma,-t-\sigma\\ m-t+1\end{matrix}\,\middle|\,-\tan^2\frac{\theta}{2}\right){}_2F_1\left(\begin{matrix}k-\sigma,-t-\sigma\\ k-t+1\end{matrix}\,\middle|\,-\tan^2\frac{\phi}{2}\right)$$
$$\times{}_2F_1\left(\begin{matrix}k-\sigma,-n-\sigma\\ k-n+1\end{matrix}\,\middle|\,\tanh^2\frac{\tau}{2}\right),$$
$$m\geqslant t,\qquad k\geqslant t,\qquad k\geqslant n; \tag{82}$$

$$\mathfrak{Z}^{\sigma}_{mn}(\cos\theta^q) = \sqrt{\frac{\Gamma(\sigma-m+1)\Gamma(\sigma+n+1)}{\Gamma(\sigma+m+1)\Gamma(\sigma-n+1)}}\cos^{2\sigma}\frac{\theta}{2}\cos^{2\sigma}\frac{\phi}{2}\cosh^{2\sigma}\frac{\tau}{2}$$
$$\times\sum_{k=-\sigma}^{\sigma}\sum_{t=-\sigma}^{\sigma}\mathbf{i}^{k-m}\tan^{t-m}\frac{\theta}{2}\tan^{k-t}\frac{\phi}{2}\tanh^{n-k}\frac{\tau}{2}$$
$$\times{}_2F_1\left(\begin{matrix}t-\sigma,-m-\sigma\\ t-m+1\end{matrix}\,\middle|\,-\tan^2\frac{\theta}{2}\right){}_2F_1\left(\begin{matrix}k-\sigma,-t-\sigma\\ k-t+1\end{matrix}\,\middle|\,-\tan^2\frac{\phi}{2}\right)$$
$$\times{}_2F_1\left(\begin{matrix}n-\sigma,-k-\sigma\\ n-k+1\end{matrix}\,\middle|\,\tanh^2\frac{\tau}{2}\right),$$
$$t\geqslant m,\qquad k\geqslant t,\qquad n\geqslant k; \tag{83}$$

$$\mathfrak{Z}^{\sigma}_{mn}(\cos\theta^q) = \sqrt{\frac{\Gamma(\sigma-m+1)\Gamma(\sigma+n+1)}{\Gamma(\sigma+m+1)\Gamma(\sigma-n+1)}}\cos^{2\sigma}\frac{\theta}{2}\cos^{2\sigma}\frac{\phi}{2}\cosh^{2\sigma}\frac{\tau}{2}$$
$$\times\sum_{k=-\sigma}^{\sigma}\sum_{t=-\sigma}^{\sigma}\mathbf{i}^{2t-m-k}\tan^{t-m}\frac{\theta}{2}\tan^{t-k}\frac{\phi}{2}\tanh^{n-k}\frac{\tau}{2}$$
$$\times{}_2F_1\left(\begin{matrix}t-\sigma,-m-\sigma\\ t-m+1\end{matrix}\,\middle|\,-\tan^2\frac{\theta}{2}\right){}_2F_1\left(\begin{matrix}t-\sigma,-k-\sigma\\ t-k+1\end{matrix}\,\middle|\,-\tan^2\frac{\phi}{2}\right)$$
$$\times{}_2F_1\left(\begin{matrix}n-\sigma,-k-\sigma\\ n-k+1\end{matrix}\,\middle|\,\tanh^2\frac{\tau}{2}\right),$$
$$t\geqslant m,\qquad t\geqslant k,\qquad n\geqslant k; \tag{84}$$

$$\mathfrak{Z}^{\sigma}_{mn}(\cos\theta^q) = \sqrt{\frac{\Gamma(\sigma-m+1)\Gamma(\sigma-n+1)}{\Gamma(\sigma+m+1)\Gamma(\sigma+n+1)}}\cos^{2\sigma}\frac{\theta}{2}\cos^{2\sigma}\frac{\phi}{2}\cosh^{2\sigma}\frac{\tau}{2}$$
$$\times\sum_{k=-\sigma}^{\sigma}\sum_{t=-\sigma}^{\sigma}\mathbf{i}^{k-m}\frac{\Gamma(\sigma+k+1)}{\Gamma(\sigma-k+1)}\tan^{t-m}\frac{\theta}{2}\tan^{k-t}\frac{\phi}{2}\tanh^{k-n}\frac{\tau}{2}$$

$$\times {}_2F_1\left(\begin{matrix}t-\sigma,-m-\sigma\\ t-m+1\end{matrix}\,\middle|\,-\tan^2\frac{\theta}{2}\right){}_2F_1\left(\begin{matrix}k-\sigma,-t-\sigma\\ k-t+1\end{matrix}\,\middle|\,-\tan^2\frac{\phi}{2}\right)$$
$$\times {}_2F_1\left(\begin{matrix}k-\sigma,-n-\sigma\\ k-n+1\end{matrix}\,\middle|\,\tanh^2\frac{\tau}{2}\right),$$
$$t\geqslant m,\qquad k\geqslant t,\qquad k\geqslant n; \tag{85}$$

$$\mathfrak{Z}^{\sigma}_{mn}(\cos\theta^q)=\sqrt{\frac{\Gamma(\sigma-m+1)\Gamma(\sigma-n+1)}{\Gamma(\sigma+m+1)\Gamma(\sigma+n+1)}}\cos^{2\sigma}\frac{\theta}{2}\cos^{2\sigma}\frac{\phi}{2}\cosh^{2\sigma}\frac{\tau}{2}$$
$$\times\sum_{k=-\sigma}^{\sigma}\sum_{t=-\sigma}^{\sigma}\mathbf{i}^{2t-m-k}\frac{\Gamma(\sigma+k+1)}{\Gamma(\sigma-k+1)}\tan^{t-m}\frac{\theta}{2}\tan^{t-k}\frac{\phi}{2}\tanh^{k-n}\frac{\tau}{2}$$
$$\times {}_2F_1\left(\begin{matrix}t-\sigma,-m-\sigma\\ t-m+1\end{matrix}\,\middle|\,-\tan^2\frac{\theta}{2}\right){}_2F_1\left(\begin{matrix}t-\sigma,-k-\sigma\\ t-k+1\end{matrix}\,\middle|\,-\tan^2\frac{\phi}{2}\right)$$
$$\times {}_2F_1\left(\begin{matrix}k-\sigma,-n-\sigma\\ k-n+1\end{matrix}\,\middle|\,\tanh^2\frac{\tau}{2}\right),$$
$$t\geqslant m,\qquad t\geqslant k,\qquad k\geqslant n; \tag{86}$$

$$\mathfrak{Z}^{\sigma}_{mn}(\cos\theta^q)=\sqrt{\frac{\Gamma(\sigma+m+1)\Gamma(\sigma+n+1)}{\Gamma(\sigma-m+1)\Gamma(\sigma-n+1)}}\cos^{2\sigma}\frac{\theta}{2}\cos^{2\sigma}\frac{\phi}{2}\cosh^{2\sigma}\frac{\tau}{2}$$
$$\times\sum_{k=-\sigma}^{\sigma}\sum_{t=-\sigma}^{\sigma}\mathbf{i}^{m-k}\frac{\Gamma(\sigma-k+1)}{\Gamma(\sigma+k+1)}\tan^{m-t}\frac{\theta}{2}\tan^{t-k}\frac{\phi}{2}\tanh^{n-k}\frac{\tau}{2}$$
$$\times {}_2F_1\left(\begin{matrix}m-\sigma,-t-\sigma\\ m-t+1\end{matrix}\,\middle|\,-\tan^2\frac{\theta}{2}\right){}_2F_1\left(\begin{matrix}t-\sigma,-k-\sigma\\ t-k+1\end{matrix}\,\middle|\,-\tan^2\frac{\phi}{2}\right)$$
$$\times {}_2F_1\left(\begin{matrix}n-\sigma,-k-\sigma\\ n-k+1\end{matrix}\,\middle|\,\tanh^2\frac{\tau}{2}\right),$$
$$m\geqslant t,\qquad t\geqslant k,\qquad n\geqslant k; \tag{87}$$

$$\mathfrak{Z}^{\sigma}_{mn}(\cos\theta^q)=\sqrt{\frac{\Gamma(\sigma+m+1)\Gamma(\sigma+n+1)}{\Gamma(\sigma-m+1)\Gamma(\sigma-n+1)}}\cos^{2\sigma}\frac{\theta}{2}\cos^{2\sigma}\frac{\phi}{2}\cosh^{2\sigma}\frac{\tau}{2}$$
$$\times\sum_{k=-\sigma}^{\sigma}\sum_{t=-\sigma}^{\sigma}\mathbf{i}^{m+k-2t}\frac{\Gamma(\sigma-k+1)}{\Gamma(\sigma+k+1)}\tan^{m-t}\frac{\theta}{2}\tan^{k-t}\frac{\phi}{2}\tanh^{n-k}\frac{\tau}{2}$$
$$\times {}_2F_1\left(\begin{matrix}m-\sigma,-t-\sigma\\ m-t+1\end{matrix}\,\middle|\,-\tan^2\frac{\theta}{2}\right){}_2F_1\left(\begin{matrix}k-\sigma,-t-\sigma\\ k-t+1\end{matrix}\,\middle|\,-\tan^2\frac{\phi}{2}\right)$$
$$\times {}_2F_1\left(\begin{matrix}n-\sigma,-k-\sigma\\ n-k+1\end{matrix}\,\middle|\,\tanh^2\frac{\tau}{2}\right),$$
$$m\geqslant t,\qquad k\geqslant t,\qquad n\geqslant k. \tag{88}$$

As is known, matrix elements of finite-dimensional representations of $SO_0(1,4)$ are expressed via the functions $f(\mathfrak{q})=\mathfrak{M}^{\sigma}_{mn}(\mathfrak{q})=\mathrm{e}^{-\mathbf{i}m\varphi^q}\mathfrak{Z}^{\sigma}_{mn}(\cos\theta^q)\,\mathrm{e}^{-\mathbf{i}n\psi^q}$, where $\mathfrak{Z}^{\sigma}_{mn}(\cos\theta^q)$

is defined by (80) or (81)–(88).[9] For example, let us calculate matrices of finite-dimensional representations at $\sigma = 0, \frac{1}{2}, 1$:

$$T_0(\varphi^q, \theta^q, \psi^q) = 1, \tag{89}$$

$$T_{\frac{1}{2}}(\varphi^q, \theta^q, \psi^q) = \begin{pmatrix} \mathfrak{M}^{\frac{1}{2}}_{-\frac{1}{2}-\frac{1}{2}} & \mathfrak{M}^{\frac{1}{2}}_{\frac{1}{2}-\frac{1}{2}} \\ \mathfrak{M}^{\frac{1}{2}}_{-\frac{1}{2}\frac{1}{2}} & \mathfrak{M}^{\frac{1}{2}}_{\frac{1}{2}\frac{1}{2}} \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{\frac{\mathbf{i}}{2}\varphi^q}\,\mathfrak{Z}^{\frac{1}{2}}_{-\frac{1}{2}-\frac{1}{2}}\,\mathrm{e}^{\frac{\mathbf{i}}{2}\psi^q} & \mathrm{e}^{\frac{\mathbf{i}}{2}\varphi^q}\,\mathfrak{Z}^{\frac{1}{2}}_{-\frac{1}{2}\frac{1}{2}}\,\mathrm{e}^{-\frac{\mathbf{i}}{2}\psi^q} \\ \mathrm{e}^{-\frac{\mathbf{i}}{2}\varphi^q}\,\mathfrak{Z}^{\frac{1}{2}}_{\frac{1}{2}-\frac{1}{2}}\,\mathrm{e}^{\frac{\mathbf{i}}{2}\psi^q} & \mathrm{e}^{-\frac{\mathbf{i}}{2}\varphi^q}\,\mathfrak{Z}^{\frac{1}{2}}_{\frac{1}{2}\frac{1}{2}}\,\mathrm{e}^{-\frac{\mathbf{i}}{2}\psi^q} \end{pmatrix}$$

$$= \begin{pmatrix} \mathrm{e}^{\frac{\mathbf{i}}{2}\varphi^q}\cos\frac{\theta^q}{2}\,\mathrm{e}^{\frac{\mathbf{i}}{2}\psi^q} & \mathbf{i}\,\mathrm{e}^{\frac{\mathbf{i}}{2}\varphi^q}\sin\frac{\theta^q}{2}\,\mathrm{e}^{-\frac{\mathbf{i}}{2}\psi^q} \\ \mathbf{i}\,\mathrm{e}^{-\frac{\mathbf{i}}{2}\varphi^q}\sin\frac{\theta^q}{2}\,\mathrm{e}^{\frac{\mathbf{i}}{2}\psi^q} & \mathrm{e}^{-\frac{\mathbf{i}}{2}\varphi^q}\cos\frac{\theta^q}{2}\,\mathrm{e}^{-\frac{\mathbf{i}}{2}\psi^q} \end{pmatrix} \tag{90}$$

$$= \begin{pmatrix} \left[\cos\frac{\theta}{2}\cosh\frac{\tau}{2}\cos\frac{\phi}{2} - \sin\frac{\theta}{2}\cosh\frac{\tau}{2}\sin\frac{\phi}{2} + \mathbf{i}\sin\frac{\theta}{2}\sinh\frac{\tau}{2}\cos\frac{\phi}{2} + \mathbf{i}\cos\frac{\theta}{2}\sinh\frac{\tau}{2}\sin\frac{\phi}{2}\right]\mathrm{e}^{\frac{1}{2}(\epsilon+\varepsilon+\omega+\mathbf{i}\varphi+\mathbf{i}\psi-\mathbf{j}\chi+\mathbf{k}\varsigma)} \\ \left[\cos\frac{\theta}{2}\sinh\frac{\tau}{2}\cos\frac{\phi}{2} - \sin\frac{\theta}{2}\sinh\frac{\tau}{2}\sin\frac{\phi}{2} + \mathbf{i}\sin\frac{\theta}{2}\cosh\frac{\tau}{2}\cos\frac{\phi}{2} + \mathbf{i}\cos\frac{\theta}{2}\cosh\frac{\tau}{2}\sin\frac{\phi}{2}\right]\mathrm{e}^{\frac{1}{2}(\varepsilon+\omega-\epsilon+\mathbf{i}\psi-\mathbf{i}\varphi-\mathbf{j}\chi-\mathbf{k}\varsigma)} \\ \left[\cos\frac{\theta}{2}\sinh\frac{\tau}{2}\cos\frac{\phi}{2} - \sin\frac{\theta}{2}\sinh\frac{\tau}{2}\sin\frac{\phi}{2} + \mathbf{i}\cos\frac{\theta}{2}\cosh\frac{\tau}{2}\sin\frac{\phi}{2} + \mathbf{i}\sin\frac{\theta}{2}\cosh\frac{\tau}{2}\cos\frac{\phi}{2}\right]\mathrm{e}^{\frac{1}{2}(\epsilon-\varepsilon-\omega+\mathbf{i}\varphi-\mathbf{i}\psi+\mathbf{j}\chi+\mathbf{k}\varsigma)} \\ \left[\cos\frac{\theta}{2}\cosh\frac{\tau}{2}\cos\frac{\phi}{2} - \sin\frac{\theta}{2}\cosh\frac{\tau}{2}\sin\frac{\phi}{2} + \mathbf{i}\cos\frac{\theta}{2}\sinh\frac{\tau}{2}\sin\frac{\phi}{2} + \mathbf{i}\sin\frac{\theta}{2}\sinh\frac{\tau}{2}\cos\frac{\phi}{2}\right]\mathrm{e}^{\frac{1}{2}(-\epsilon-\varepsilon-\omega-\mathbf{i}\varphi-\mathbf{i}\psi+\mathbf{j}\chi-\mathbf{k}\varsigma)} \end{pmatrix}, \tag{91}$$

$$T_1(\varphi^q, \theta^q, \psi^q) = \begin{pmatrix} \mathfrak{M}^1_{-1-1} & \mathfrak{M}^1_{-10} & \mathfrak{M}^1_{-11} \\ \mathfrak{M}^1_{0-1} & \mathfrak{M}^1_{00} & \mathfrak{M}^1_{01} \\ \mathfrak{M}^1_{1-1} & \mathfrak{M}^1_{10} & \mathfrak{M}^1_{11} \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{\mathbf{i}\varphi^q}\,\mathfrak{Z}^1_{-1-1}\,\mathrm{e}^{\mathbf{i}\psi^q} & \mathrm{e}^{\mathbf{i}\varphi^q}\,\mathfrak{Z}^1_{-10} & \mathrm{e}^{\mathbf{i}\varphi^q}\,\mathfrak{Z}^1_{-11}\,\mathrm{e}^{-\mathbf{i}\psi^q} \\ \mathfrak{Z}^1_{0-1}\,\mathrm{e}^{\mathbf{i}\psi^q} & \mathfrak{Z}^1_{00} & \mathfrak{Z}^1_{01}\,\mathrm{e}^{-\mathbf{i}\psi^q} \\ \mathrm{e}^{-\mathbf{i}\varphi^q}\,\mathfrak{Z}^1_{1-1}\,\mathrm{e}^{\mathbf{i}\psi^q} & \mathrm{e}^{-\mathbf{i}\varphi^q}\,\mathfrak{Z}^1_{10} & \mathrm{e}^{-\mathbf{i}\varphi^q}\,\mathfrak{Z}^1_{11}\,\mathrm{e}^{-\mathbf{i}\psi^q} \end{pmatrix}$$

$$= \begin{pmatrix} \mathrm{e}^{\mathbf{i}\varphi^q}\cos^2\frac{\theta^q}{2}\,\mathrm{e}^{\mathbf{i}\psi^q} & \frac{\mathbf{i}}{\sqrt{2}}\,\mathrm{e}^{\mathbf{i}\varphi^q}\sin\theta^q & -\mathrm{e}^{\mathbf{i}\varphi^q}\sin^2\frac{\theta^q}{2}\,\mathrm{e}^{-\mathbf{i}\psi^q} \\ \frac{\mathbf{i}}{\sqrt{2}}\sin\theta^q\,\mathrm{e}^{\mathbf{i}\psi^q} & \cos\theta^q & \frac{\mathbf{i}}{\sqrt{2}}\sin\theta^q\,\mathrm{e}^{-\mathbf{i}\psi^q} \\ -\mathrm{e}^{-\mathbf{i}\varphi^q}\sin^2\frac{\theta^q}{2}\,\mathrm{e}^{\mathbf{i}\psi^q} & \frac{\mathbf{i}}{\sqrt{2}}\,\mathrm{e}^{-\mathbf{i}\varphi^q}\sin\theta^q & \mathrm{e}^{-\mathbf{i}\varphi^q}\cos^2\frac{\theta^q}{2}\,\mathrm{e}^{-\mathbf{i}\psi^q} \end{pmatrix}, \tag{92}$$

where

$$\sin\theta^q = \sin\theta\cos\phi\cosh\tau + \cos\theta\sin\phi\cosh\tau - \mathbf{i}\cos\theta\cos\phi\sinh\tau + \mathbf{i}\sin\theta\sin\phi\sinh\tau,$$

$$\cos\theta^q = \cos\theta\cos\phi\cosh\tau - \sin\theta\sin\phi\cosh\tau + \mathbf{i}\cos\theta\sin\phi\sinh\tau + \mathbf{i}\sin\theta\cos\phi\sinh\tau,$$

$$\sin^2\frac{\theta^q}{2} = \sin^2\frac{\theta}{2}\cos^2\frac{\phi}{2}\cosh^2\frac{\tau}{2} + \frac{1}{2}\sin\theta\sin\phi\cosh\tau + \cos^2\frac{\theta}{2}\sin^2\frac{\phi}{2}\cosh^2\frac{\tau}{2} - \frac{\mathbf{i}}{2}\left(\sin\frac{\theta}{2}\cos\frac{\phi}{2} + \cos\frac{\theta}{2}\sin\frac{\phi}{2}\right)\sinh\tau - \cos^2\frac{\theta}{2}\cos^2\frac{\phi}{2}\sinh^2\frac{\tau}{2} - \sin^2\frac{\theta}{2}\sin^2\frac{\phi}{2}\sinh^2\frac{\tau}{2},$$

$$\cos^2\frac{\theta^q}{2} = \cos^2\frac{\theta}{2}\cos^2\frac{\phi}{2}\cosh^2\frac{\tau}{2} - \frac{1}{2}\sin\theta\sin\phi\cosh\tau + \sin^2\frac{\theta}{2}\sin^2\frac{\phi}{2}\cosh^2\frac{\tau}{2} + \frac{\mathbf{i}}{2}\left(\sin\frac{\theta}{2}\cos\frac{\phi}{2} + \cos\frac{\theta}{2}\sin\frac{\phi}{2}\right)\sinh\tau - \sin^2\frac{\theta}{2}\cos^2\frac{\phi}{2}\sinh^2\frac{\tau}{2} - \cos^2\frac{\theta}{2}\sin^2\frac{\phi}{2}\sinh^2\frac{\tau}{2}.$$

It is easy to see that $T_{\frac{1}{2}}(\varphi^q, \theta^q, \psi^q)$ is the fundamental representation (18) of $Sp(1, 1)$.

[9] The functions $f(\mathfrak{q}) = \mathfrak{M}^{\sigma}_{mn}(\mathfrak{q})$ are eigenfunctions of the Laplace–Beltrami operator $\triangle_L(\mathfrak{S}_{10}) = -F$ defined on the group manifold $\mathfrak{S}_{10}$ of $SO_0(1, 4)$. An explicit expression for $\triangle_L(\mathfrak{S}_{10}) = -F$ is given by formula (42).

Spherical functions of the second type $f(\varphi^q,\theta^q)=\mathfrak{M}^m_\sigma(\varphi^q,\theta^q,0)=\mathrm{e}^{-im\varphi^q}\mathfrak{Z}^m_\sigma(\cos\theta^q)$, where

$$\mathfrak{Z}^m_\sigma(\cos\theta^q)=\sum_{k=-\sigma}^{\sigma}\sum_{t=-\sigma}^{\sigma}P^\sigma_{mt}(\cos\theta)P^\sigma_{tk}(\cos\phi)\mathfrak{P}^k_\sigma(\cosh\tau)$$

is *an associated hyperspherical function* of $SO_0(1,4)$, are defined on the surface of the quaternion 2-sphere S^2_q. $\mathfrak{Z}^m_\sigma(\cos\theta^q)$ are eigenfunctions of the Laplace–Beltrami operator $\triangle_L\big(S^2_q\big)=-F$,

$$\triangle_L\big(S^2_q\big)=\frac{\partial^2}{\partial\theta^{q\,2}}+\cot\theta^q\frac{\partial}{\partial\theta^q}+\frac{1}{\sin^2\theta^q}\frac{\partial^2}{\partial\varphi^{q\,2}}.$$

Hypergeometric-type formulae for $\mathfrak{Z}^m_\sigma(\cos\theta^q)$ are

$$\begin{aligned}\mathfrak{Z}^m_\sigma(\cos\theta^q)&=\sqrt{\frac{\Gamma(\sigma+m+1)}{\Gamma(\sigma-m+1)}}\cos^{2\sigma}\frac{\theta}{2}\cos^{2\sigma}\frac{\phi}{2}\cosh^{2\sigma}\frac{\tau}{2}\\&\times\sum_{k=-\sigma}^{\sigma}\sum_{t=-\sigma}^{\sigma}\mathbf{i}^{m-k}\tan^{m-t}\frac{\theta}{2}\tan^{t-k}\frac{\phi}{2}\tanh^k\frac{\tau}{2}\\&\times{}_2F_1\left(\begin{matrix}m-\sigma,-t-\sigma\\m-t+1\end{matrix}\middle|-\tan^2\frac{\theta}{2}\right){}_2F_1\left(\begin{matrix}t-\sigma,-k-\sigma\\t-k+1\end{matrix}\middle|-\tan^2\frac{\phi}{2}\right)\\&\times{}_2F_1\left(\begin{matrix}k-\sigma,-\sigma\\k+1\end{matrix}\middle|\tanh^2\frac{\tau}{2}\right),\\&m\geqslant t,\quad t\geqslant k;\end{aligned}$$

$$\begin{aligned}\mathfrak{Z}^m_\sigma(\cos\theta^q)&=\sqrt{\frac{\Gamma(\sigma+m+1)}{\Gamma(\sigma-m+1)}}\cos^{2\sigma}\frac{\theta}{2}\cos^{2\sigma}\frac{\phi}{2}\cosh^{2\sigma}\frac{\tau}{2}\\&\times\sum_{k=-\sigma}^{\sigma}\sum_{t=-\sigma}^{\sigma}\mathbf{i}^{m+k-2t}\tan^{m-t}\frac{\theta}{2}\tan^{k-t}\frac{\phi}{2}\tanh^k\frac{\tau}{2}\\&\times{}_2F_1\left(\begin{matrix}m-\sigma,-t-\sigma\\m-t+1\end{matrix}\middle|-\tan^2\frac{\theta}{2}\right){}_2F_1\left(\begin{matrix}k-\sigma,-t-\sigma\\k-t+1\end{matrix}\middle|-\tan^2\frac{\phi}{2}\right)\\&\times{}_2F_1\left(\begin{matrix}k-\sigma,-\sigma\\k+1\end{matrix}\middle|\tanh^2\frac{\tau}{2}\right),\\&m\geqslant t,\quad k\geqslant t;\end{aligned}$$

$$\begin{aligned}\mathfrak{Z}^m_\sigma(\cos\theta^q)&=\sqrt{\frac{\Gamma(\sigma-m+1)}{\Gamma(\sigma+m+1)}}\cos^{2\sigma}\frac{\theta}{2}\cos^{2\sigma}\frac{\phi}{2}\cosh^{2\sigma}\frac{\tau}{2}\\&\times\sum_{k=-\sigma}^{\sigma}\sum_{t=-\sigma}^{\sigma}\mathbf{i}^{k-m}\frac{\Gamma(\sigma+k+1)}{\Gamma(\sigma-k+1)}\tan^{t-m}\frac{\theta}{2}\tan^{k-t}\frac{\phi}{2}\tanh^k\frac{\tau}{2}\\&\times{}_2F_1\left(\begin{matrix}t-\sigma,-m-\sigma\\t-m+1\end{matrix}\middle|-\tan^2\frac{\theta}{2}\right){}_2F_1\left(\begin{matrix}k-\sigma,-t-\sigma\\k-t+1\end{matrix}\middle|-\tan^2\frac{\phi}{2}\right)\\&\times{}_2F_1\left(\begin{matrix}k-\sigma,-\sigma\\k+1\end{matrix}\middle|\tanh^2\frac{\tau}{2}\right),\\&t\geqslant m,\quad k\geqslant t;\end{aligned}$$

$$\mathfrak{Z}^m_\sigma(\cos\theta^q) = \sqrt{\frac{\Gamma(\sigma-m+1)}{\Gamma(\sigma+m+1)}}\cos^{2\sigma}\frac{\theta}{2}\cos^{2\sigma}\frac{\phi}{2}\cosh^{2\sigma}\frac{\tau}{2}$$
$$\times\sum_{k=-\sigma}^{\sigma}\sum_{t=-\sigma}^{\sigma}\mathbf{i}^{2t-m-k}\frac{\Gamma(\sigma+k+1)}{\Gamma(\sigma-k+1)}\tan^{t-m}\frac{\theta}{2}\tan^{t-k}\frac{\phi}{2}\tanh^{k}\frac{\tau}{2}$$
$$\times\,{}_2F_1\left(\begin{matrix}t-\sigma,-m-\sigma\\ t-m+1\end{matrix}\,\middle|\,-\tan^2\frac{\theta}{2}\right){}_2F_1\left(\begin{matrix}t-\sigma,-k-\sigma\\ t-k+1\end{matrix}\,\middle|\,-\tan^2\frac{\phi}{2}\right)$$
$$\times\,{}_2F_1\left(\begin{matrix}k-\sigma,-\sigma\\ k+1\end{matrix}\,\middle|\,\tanh^2\frac{\tau}{2}\right),$$
$$t\geqslant m,\quad t\geqslant k.$$

The latter formulae hold at any k when σ is an half-integer number. When σ is an integer number, these formulae hold at $k=0,1,\ldots,\sigma-1,\sigma$. At $k=-\sigma,-\sigma+1,\ldots,0$ we must replace the function

$${}_2F_1\left(\begin{matrix}k-\sigma,-\sigma\\ k+1\end{matrix}\,\middle|\,\tanh^2\frac{\tau}{2}\right)\text{ via }{}_2F_1\left(\begin{matrix}-\sigma,-k-\sigma\\ -k+1\end{matrix}\,\middle|\,\tanh^2\frac{\tau}{2}\right)$$

and $\tanh^k\frac{\tau}{2}$ via $\tanh^{-k}\frac{\tau}{2}$.

At $m=n=0$ we obtain *a zonal hyperspherical function* $\mathfrak{Z}_\sigma(\cos\theta^q)\equiv\mathfrak{Z}^\sigma_{00}(\cos\theta^q)$ of the group $SO_0(1,4)$. Namely,

$$\mathfrak{Z}_\sigma(\cos\theta^q) = \cos^{2\sigma}\frac{\theta}{2}\cos^{2\sigma}\frac{\phi}{2}\cosh^{2\sigma}\frac{\tau}{2}$$
$$\times\sum_{k=-\sigma}^{\sigma}\sum_{t=-\sigma}^{\sigma}\mathbf{i}^{k}\frac{\Gamma(\sigma+k+1)}{\Gamma(\sigma-k+1)}\tan^{t}\frac{\theta}{2}\tan^{k-t}\frac{\phi}{2}\tanh^{k}\frac{\tau}{2}$$
$$\times\,{}_2F_1\left(\begin{matrix}t-\sigma,-\sigma\\ t+1\end{matrix}\,\middle|\,-\tan^2\frac{\theta}{2}\right){}_2F_1\left(\begin{matrix}k-\sigma,-t-\sigma\\ k-t+1\end{matrix}\,\middle|\,-\tan^2\frac{\phi}{2}\right)$$
$$\times\,{}_2F_1\left(\begin{matrix}k-\sigma,-\sigma\\ k+1\end{matrix}\,\middle|\,\tanh^2\frac{\tau}{2}\right),$$
$$k\geqslant t;$$

$$\mathfrak{Z}_\sigma(\cos\theta^q) = \cos^{2\sigma}\frac{\theta}{2}\cos^{2\sigma}\frac{\phi}{2}\cosh^{2\sigma}\frac{\tau}{2}$$
$$\times\sum_{k=-\sigma}^{\sigma}\sum_{t=-\sigma}^{\sigma}\mathbf{i}^{-k}\frac{\Gamma(\sigma-k+1)}{\Gamma(\sigma+k+1)}\tan^{-t}\frac{\theta}{2}\tan^{t-k}\frac{\phi}{2}\tanh^{-k}\frac{\tau}{2}$$
$$\times\,{}_2F_1\left(\begin{matrix}-\sigma,-t-\sigma\\ -t+1\end{matrix}\,\middle|\,-\tan^2\frac{\theta}{2}\right){}_2F_1\left(\begin{matrix}t-\sigma,-k-\sigma\\ t-k+1\end{matrix}\,\middle|\,-\tan^2\frac{\phi}{2}\right)$$
$$\times\,{}_2F_1\left(\begin{matrix}-\sigma,-k-\sigma\\ -k+1\end{matrix}\,\middle|\,\tanh^2\frac{\tau}{2}\right),$$
$$t\geqslant k.$$

In its turn, the functions $f(\dot\varphi^q,\dot\theta^q)=\mathrm{e}^{\mathbf{i}m\dot\varphi^q}\mathfrak{Z}^m_\sigma(\cos\dot\theta^q)$ (or $f(\dot\theta^q)=\mathfrak{Z}_\sigma(\cos\dot\theta^q)$) are defined on the surface of the dual quaternion sphere S^2_q. Explicit expressions and hypergeometric-type formulae for $f(\dot\varphi^q,\dot\theta^q)$ are analogous to the previous expressions for $f(\varphi^q,\theta^q)$.

Spherical functions of the fourth type $f(\varphi,\theta,\psi) = \mathfrak{M}^{\sigma}_{mn}(\varphi,\theta,\psi) = \mathrm{e}^{-\mathrm{i}m\varphi}P^{\sigma}_{mn}(\cos\theta)\,\mathrm{e}^{-\mathrm{i}n\psi}$ (or $f(\varsigma,\phi,\chi) = \mathfrak{M}^{\sigma}_{mn}(\varsigma,\phi,\chi) = \mathrm{e}^{-\mathrm{i}m\varsigma}P^{\sigma}_{mn}(\cos\phi)\,\mathrm{e}^{-\mathrm{i}n\chi}$) are defined on the surface of the real 3-sphere $S^3 = SO(4)/SO(3)$. Let $L^2(S^3)$ be a Hilbert space of the functions defined on the sphere S^3 in the space $\mathbb{R}^4$. Since $S^3 \sim SO_0(1,4)/P \sim K/M$, then the representations of the principal non-unitary (spherical) series $T_{\omega\sigma}$ are defined by the complex number σ and an irreducible unitary representation ω of the subgroup $M = SO(3)$. Thus, representations of the group $SO_0(1,4)$, which have a class 1 with respect to $K = SO(4)$, are realized in the space $L^2(S^3)$. At this point, spherical functions of the representations Q^m of $SO(4)$ form an orthogonal basis in $L^2(S^3)$. Therefore, we have here representations of $SO_0(1,4)$ restricted to the subgroup $SO(4)$:

$$\hat{T}^{\sigma}\downarrow^{SO_0(1,4)}_{SO(4)} = \sum_{m=0}^{l}\oplus Q^m.$$

4.1. Spherical functions on the hyperboloid and their applications to hydrogen atom problem

In 1935, using a stereographic projection of the momentum space onto a four-dimensional sphere, Fock showed [12] that Schrödinger equation for hydrogen atom is transformed into an integral equation for hyperspherical functions defined on the surface of the four-dimensional sphere. This discovery elucidates an intrinsic nature of an additional degeneration of the energy levels in hydrogen atom, and also it allows one to write important relations for wavefunctions (for example, Fock wrote simple expressions for the density matrix of the system of wavefunctions for energy levels with an arbitrary quantum number n). In 1968, authors of the work [5] showed that Fock integral equation can be written in the form of a Klein–Gordon-type equation for spherical functions defined on the surface of the four-dimensional hyperboloid. 'Square roots' of the Klein–Gordon-type equation are Dirac-like equations (in the paper [5] these equations are called Majorana-type equations) or more general Gel'fand–Yaglom-type equations [13]. Equations of this type were first considered by Dirac in 1935 [8]. Here there is an analogy with the usual formulation of the Dirac equation for a hydrogen atom in the Minkowski spacetime, but the main difference lies in the fact that Dirac-like equations are defined on the four-dimensional hyperboloid[10] immersed into a five-dimensional de Sitter space.

So, spherical functions of the third type $f(\epsilon,\tau,\varepsilon,\omega) = \mathfrak{M}^{\sigma}_{mn}(\epsilon,\tau,\varepsilon,\omega) = \mathrm{e}^{-m\epsilon}\mathfrak{P}^{\sigma}_{mn}(\cosh\tau)\,\mathrm{e}^{-n(\varepsilon+\omega)}$ are defined on the upper sheet H^4_+ of the four-dimensional hyperboloid $[\mathbf{x},\mathbf{x}] = 1$, where $\mathfrak{P}^{\sigma}_{mn}(\cosh\tau)$ is a Jacobi function[11] considered in details by Vilenkin [31]. The functions $\mathfrak{M}^{\sigma}_{mn}(\epsilon,\tau,\varepsilon,\omega)$ are eigenfunctions of the Laplace–Beltrami operator $\triangle_L\big(H^4_+\big) = -F$ defined on H^4_+:

$$\big[\triangle_L\big(H^4_+\big) - \sigma(\sigma+3)\big]\mathfrak{M}^{\sigma}_{mn}(\epsilon,\tau,\varepsilon,\omega) = 0,$$

where

$$\triangle_L\big(H^4_+\big) = -\frac{\partial^2}{\partial\tau^2} - \coth\tau\frac{\partial}{\partial\tau} - \frac{1}{\sinh^2\tau}\left[\frac{\partial^2}{\partial\epsilon^2} - 2\cosh\tau\frac{\partial^2}{\partial\epsilon\partial(\varepsilon+\omega)} + \frac{\partial^2}{\partial(\varepsilon+\omega)^2}\right],$$

or

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}\tau^2} - \coth\tau\frac{\mathrm{d}}{\mathrm{d}\tau} + \frac{m^2+n^2-2mn\cosh\tau}{\sinh^2\tau} - \sigma(\sigma+3)\right]\mathfrak{P}^{\sigma}_{mn}(\cosh\tau) = 0.$$

[10] As is known, this hyperboloid can be understood as the four-dimensional Minkowski spacetime endowed globally with a constant negative curvature.

[11] Representations of the group $SU(1,1) \simeq SL(2,\mathbb{R})$, known also as a three-dimensional Lorentz group, are expressed via the functions $\mathfrak{P}^{\sigma}_{mn}(\cosh\tau)$.

After substitution $y = \cosh\tau$ this equation can be rewritten as

$$\left[(y^2-1)\frac{\mathrm{d}^2}{\mathrm{d}y^2} + 2y\frac{\mathrm{d}}{\mathrm{d}y} - \frac{m^2+n^2-2mny}{y^2-1} + \sigma(\sigma+3)\right]\mathfrak{P}^{\sigma}_{mn}(y) = 0.$$

Let us construct a quasiregular representation of the group $SO_0(1,4)$ on the functions $f(x)$ from H^4_+, where $x = (\epsilon, \tau, \varepsilon, \omega)$. Let $L^2\big(H^4_+\big)$ be a Hilbert space of the functions on the hyperboloid H^4_+ with a scalar product

$$\begin{aligned}\langle f_1, f_2\rangle &= \int_{H^4_+} \overline{f_1(x)} f_2(x)\,\mathrm{d}\mu(x)\\ &= \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \overline{\mathfrak{P}^{\sigma_1}_{mn}(\cosh\tau)}\mathfrak{P}^{\sigma_2}_{mn}(\cosh\tau)\,\mathrm{e}^{-2m\epsilon-2n(\omega+\varepsilon)}\sinh\tau\,\mathrm{d}\tau\,\mathrm{d}\epsilon\,\mathrm{d}\varepsilon\,\mathrm{d}\omega,\end{aligned}$$

where $\mathrm{d}\mu(x)$ is an invariant measure on H^4_+ with respect to $SO_0(1,4)$. This measure is defined by an equality $\mathrm{d}\mu(x) = \sinh\tau\,\mathrm{d}\tau\,\mathrm{d}\epsilon\,\mathrm{d}\varepsilon\,\mathrm{d}\omega$. In accordance with (12) the range of variables ϵ, τ, ε, ω is $(-\infty, +\infty)$, but we consider here the upper sheet of the hyperboloid; therefore, the range of these variables is $(0, \infty)$. *A quasiregular representation* T in the space $L^2\big(H^4_+\big)$ is defined by the formula

$$T(\mathfrak{q})f(x) = f(\mathfrak{q}^{-1}x), \qquad x \in H^4_+.$$

It is easy to show that this representation is unitary. However, T is reducible, and in accordance with Gel'fand–Graev theorem [14] is decomposed into a direct integral of irreducible representations T^σ of the principal unitary series ($\sigma = -3/2 + \mathrm{i}\rho, 0 < \rho < \infty$).

Analogously, a quasiregular representation of the group $SO_0(1,4)$ in a Hilbert space $L^2\big(C^4_+\big)$ of the functions on the upper sheet C^4_+ of the cone C^4 $\big(C^4_+\colon\ x_0^2-x_1^2-x_2^2-x_3^2-x_4^2 = 0,$ $x_0 > 0\big)$ has the following form:

$$T(g)f(x) = f(g^{-1}x), \qquad x \in C^4_+.$$

This representation is unitary with respect to a scalar product

$$\langle f_1, f_2\rangle = \int_{C^4_+} \overline{f_1(x)} f_2(x)\,\mathrm{d}\mu(x)$$

defined on $L^2\big(C^4_+\big)$. Here, $\mathrm{d}\mu(x)$ is an invariant measure on C^4_+ with respect to $SO_0(1,4)$. This representation is reducible. Irreducible unitary representations of the group $SO_0(1,4)$ can be constructed in a Hilbert space of homogeneous functions on the cone [31].

Let us consider applications of the spherical functions $f(\epsilon, \tau, \varepsilon, \omega)$ to hydrogen and antihydrogen atom problems (about antihydrogen atom, see [10]). As it has been shown in the work [5] when the internal motion can be described by algebraic methods, as in the case of hydrogen atom, the proposed equation for the motion of the system as a whole (motion of the c.m.) is equivalent to a Majorana-type equation, free from the well-known difficulties such as a spacelike solution. As is known, the Bethe–Salpeter equation for two spinors of masses m_1 and m_2,

$$(\hat{p}_1 - m_1)(\hat{p}_2 - m_2)\psi(p_1, p_2) = \frac{\mathbf{i}}{2\pi}\iint G(p_1, p_2; p_1', p_2')\psi(p_1', p_2')\,\mathrm{d}p_1'\,\mathrm{d}p_2',$$

in the ladder approximation can be written as follows:

$$(c_1\hat{P}^{(1)} + \hat{p}^{(1)} - m_1)(c_2\hat{P}^{(2)} - \hat{p}^{(2)} - m_2)\psi_P(p) = \frac{\mathbf{i}}{2\pi}\int G(q)\psi_P(p+q)\,\mathrm{d}q,$$

where

$$P = p_1 + p_2, \qquad p = c_2 p_1 - c_1 p_2,$$
$$c_1 = m_1/(m_1 + m_2), \qquad c_2 = m_2/(m_1 + m_2),$$

the metric is $g_{\mu\nu} = +1, -1, -1, -1$, and the superscripts on $\hat{P}^{(i)}$ and $\hat{p}^{(i)}$ refer to the γ matrices. In this case, projection operators can be defined as

$$\Lambda_{\pm}^{(i)} = [\mathcal{E}_i(p) \pm \mathcal{K}_i]/2\mathcal{E}_i(p), \tag{93}$$

where

$$\mathcal{E}_i = \left[P^2\left(m_i^2 - p^2\right) + (p \cdot P)^2\right]^{1/2},$$
$$\mathcal{K}_1 = \left[m_1 \hat{P}^{(1)} - \mathbf{i} P^\mu \sigma_{\mu\nu}^{(1)} p^\nu\right], \qquad \mathcal{K}_2 = \left[m_1 \hat{P}^{(2)} + \mathbf{i} P^\mu \sigma_{\mu\nu}^{(2)} p^\nu\right]$$

with

$$\sigma_{\mu\nu}^{(i)} = (1/2\mathbf{i})\left[\gamma_\mu^{(i)}, \gamma_\nu^{(i)}\right].$$

Further, using the operators (93), we obtain

$$\left(P^2 - \mathcal{K}_1^2 - \mathcal{K}_2^2\right)\varphi(p^T) = -\left(\Lambda_+^{(1)}\Lambda_+^{(2)} - \Lambda_-^{(1)}\Lambda_-^{(2)}\right)\hat{P}^{(1)}\hat{P}^{(2)} \int G(p^T - l)\varphi(l)\delta(l \cdot p)\,\mathrm{d}l, \tag{94}$$

where $p_\mu^T = p_\mu - p^L u_\mu$ is the transverse relative momenta, and $p^L = p \cdot u$, $u^\mu = P^\mu/|P|$, $\phi(l) = \int_{-\infty}^{+\infty} \psi(l, q^L)\,\mathrm{d}q^L$. The approximation

$$\Lambda_+^{(1)}\Lambda_+^{(2)} - \Lambda_-^{(1)}\Lambda_-^{(2)} = +1$$

means that we take only positive-energy states for the constituents. On the other hand, the choice

$$\Lambda_+^{(1)}\Lambda_+^{(2)} - \Lambda_-^{(1)}\Lambda_-^{(2)} = -1$$

would have meant taking only negative-energy states for the system and would correspond to charge conjugation for the c.m. motion.

Since $\Lambda_+^i = 1$ is equivalent to

$$\mathcal{K}_i = \mathcal{E}_i = \left(m_i^2 - (p^T)^2\right)^{1/2}|P|,$$

then equation (94) can be written as

$$[P^2 - |P|(m_1 + m_2 - (p^T)^2/2\mu)]\varphi(p^T) = P^2 \int G(p^T - l)\varphi(l)\delta(l \cdot P)\,\mathrm{d}l,$$

where

$$\mu = \frac{m_1 m_2}{m_1 + m_2}.$$

In the case of hydrogen atom this equation has the form

$$[|P| - (m_1 + m_2 - (p^T)^2/2\mu)]\varphi(p^T) = |P|\frac{\mathrm{e}^2}{2\pi} \int \frac{1}{(p^T - l)^2}\delta(l \cdot P)\varphi(l)\,\mathrm{d}l. \tag{95}$$

Using the Fock stereographic projection [4, 12]

$$\xi_\mu = 2a p_\mu (a^2 - p^2), \qquad \xi_4 = (a^2 + p^2)/(a^2 - p^2), \qquad \mu = 0, \ldots, 3,$$

where $p^2 = p_\mu p^\mu$ and a is an arbitrary constant, we will project stereographically the four-dimensional p-space on a five-dimensional hyperboloid. This projection allows us to rewrite equation (95) in the form of a Klein–Gordon-type equation

$$(P^2 - \mathcal{K}^2)\Psi_P = 0 \tag{96}$$

with

$$\mathcal{K} = m_1 + m_2 - \mu\, \mathrm{e}^4/2N,$$

and N^2 is the operator $D^T + 1$, where D^T is the angular part of the four-dimensional Laplace operator. $\Psi_P(\xi_a)$ form a basis for the representation of the de Sitter group $SO_0(1,4)$. A 'square root' of the Klein–Gordon-type equation (96) is a Majorana-type equation

$$[\Gamma \cdot P - (m_1 + m_2)N + \mathrm{e}^4\, \mu/2N]\Psi_P = 0 \tag{97}$$

or

$$[\Gamma \cdot P + (m_1 + m_2)N - \mathrm{e}^4\, \mu/2N]\dot{\psi}_P = 0 \tag{98}$$

where Γ-matrices behave like components of a 5-vector in $SO_0(1,4)$. Equations (97) and (98) describe hydrogen and antihydrogen atoms, respectively.

In equations (96)–(98), the functions Ψ_P are eigenfunctions of the Laplace–Beltrami operator defined on the surface of the five-dimensional hyperboloid (more precisely speaking, on the upper sheet H^4_+ of this hyperboloid for equation (97) and on the lower sheet H^4_- for (98)). As it has been shown previously, this hyperboloid is a homogeneous space of the de Sitter group $SO_0(1,4)$. On the other hand, spherical functions Ψ_p are solutions of equations (96)–(98), that is, they are wavefunctions, and for that reason Ψ_P play a crucial role in the hydrogen (antihydrogen) atom problem.

Let us consider in brief solutions (wavefunctions) of the Majorana-type equations (97) and (98). To this end in view we must introduce an *inhomogeneous de Sitter group* $ISO_0 = SO_0(1,4) \odot T_5$, which is a semidirect product of the subgroup $SO_0(1,4)$ (connected component) of five-dimensional rotations and a subgroup T_5 of five-dimensional translations of the de Sitter space $\mathbb{R}^{1,4}$. The subgroup T_5 is a direct product of five one-dimensional translation groups T_1, $T_5 = T_1 \otimes T_1 \otimes T_1 \otimes T_1 \otimes T_1$. At this point, each group T_1 is isomorphic to the group $\mathbb{R}^+$ of all positively defined real numbers. At the restriction to H^4_+, the maximal homogeneous space $\mathcal{M}_{15} = \mathbb{R}^{1,4} \times \mathfrak{S}_{10}$ of $ISO_0(1,4)$ is reduced to $\mathcal{M}_9 = \mathbb{R}^{1,4} \times H^4_+$. Let $F(x, \epsilon, \tau, \varepsilon, \omega)$ be a square integrable function on $\mathcal{M}_9$, that is,

$$\int_{H^4_+}\int_{T_5} |F|^2\, \mathrm{d}^5x\, \mathrm{d}^4g < +\infty,$$

then in the case of finite-dimensional representations of $SO_0(1,4)$ there is an expansion of $F(x, \epsilon, \tau, \varepsilon, \omega)$ in a Fourier-type integral

$$F(x, \epsilon, \tau, \varepsilon, \omega) = \sum_{\sigma=0}^{\infty} \sum_{m,n=-\sigma}^{\sigma} \int_{T_5} \alpha^\sigma_{mn}\, \mathrm{e}^{\mathrm{i}px}\, \mathrm{e}^{-m\epsilon - n(\varepsilon+\omega)}\, \mathfrak{P}^\sigma_{mn}(\cosh\tau)\, \mathrm{d}^5x, \tag{99}$$

where

$$\alpha^\sigma_{mn} = \frac{(-1)^{m-n}(2\sigma+3)}{16\pi^2} \int_{H^4_+}\int_{T_5} F\, \mathrm{e}^{-\mathrm{i}px}\, \mathfrak{P}^\sigma_{mn}(\cosh\tau)\, \mathrm{e}^{-m\epsilon - n(\varepsilon+\omega)}\, \mathrm{d}^5x\, \mathrm{d}^4g,$$

and $\mathrm{d}^4g = \sinh\tau\, \mathrm{d}\tau\, \mathrm{d}\epsilon\, \mathrm{d}\varepsilon\, \mathrm{d}\omega$ is a Haar measure on the hyperboloid H^4_+.

Further, let T be an unbounded region in $\mathbb{R}^{1,4}$ and let Σ be a surface of the hyperboloid H^4_+ (correspondingly, $\dot{\Sigma}$, for the sheet H^4_-), then it needs to find a function $\psi(g) = \left(\psi^m_P(g), \dot{\psi}^{\dot{m}}_P(g)\right)^T$ in the all region T. $\psi(g)$ is a continuous function (everywhere in T), including the surfaces Σ and $\dot{\Sigma}$. At this point, $\psi^m_P(g)|_\Sigma = F_m(g)$, $\dot{\psi}^{\dot{m}}_P(g)\big|_{\dot{\Sigma}} = \dot{F}_{\dot{m}}(g)$, where $F_m(g)$ and $\dot{F}_{\dot{m}}(g)$ are square integrable functions (boundary conditions) defined on the surfaces Σ and $\dot{\Sigma}$, respectively.

Following the method proposed in [24, 25, 26, 28], we can find solutions of the boundary value problem in the form of Fourier-type series

$$\psi^m_P = \sum_{\sigma=0}^{\infty} \sum_k \boldsymbol{f}_{\sigma mk}(r) \sum_{n=-\sigma}^{\sigma} \boldsymbol{\alpha}^m_{\sigma n} \mathfrak{M}^\sigma_{mn}(\epsilon, \tau, \varepsilon, \omega), \tag{100}$$

$$\dot{\psi}^{\dot{m}}_P(g) = \sum_{\dot{\sigma}=0}^{\infty} \sum_{\dot{k}} \overset{*}{\boldsymbol{f}}_{\dot{\sigma}\dot{m}\dot{k}}(r^*) \sum_{\dot{n}=-\dot{\sigma}}^{\dot{\sigma}} \boldsymbol{\alpha}^{\dot{m}}_{\dot{\sigma}\dot{n}} \overset{*}{\mathfrak{M}}{}^{\dot{\sigma}}_{\dot{m}\dot{n}}(\epsilon, \tau, \varepsilon, \omega), \tag{101}$$

where

$$\boldsymbol{\alpha}^m_{\sigma n} = \frac{(-1)^n(2\sigma+3)}{16\pi^2} \int_{H^4_+} F_m \mathfrak{M}^\sigma_{mn}(\epsilon, \tau, \varepsilon, \omega) \sinh\tau \, \mathrm{d}\tau \, \mathrm{d}\epsilon \, \mathrm{d}\varepsilon \, \mathrm{d}\omega,$$

$$\boldsymbol{\alpha}^{\dot{m}}_{\dot{\sigma}\dot{n}} = \frac{(-1)^{\dot{n}}(2\dot{\sigma}+3)}{16\pi^2} \int_{H^4_-} F_{\dot{m}} \overset{*}{\mathfrak{M}}{}^{\dot{\sigma}}_{\dot{m}\dot{n}}(\epsilon, \tau, \varepsilon, \omega) \sinh\tau \, \mathrm{d}\tau \, \mathrm{d}\epsilon \, \mathrm{d}\varepsilon \, \mathrm{d}\omega.$$

The indices k and $\dot{k}$ numerate equivalent representations. $\mathfrak{M}^\sigma_{mn}(\epsilon, \tau, \varepsilon, \omega)$ $\left(\overset{*}{\mathfrak{M}}{}^{\dot{\sigma}}_{\dot{m}\dot{n}}(\epsilon, \tau, \varepsilon, \omega)\right)$ are hyperspherical functions defined on the surface Σ ($\dot{\Sigma}$) of the four-dimensional hyperboloid H^4 of the radius r (r^*) (H^4 can be understood as a four-dimensional sphere with an imaginary radius r), $\boldsymbol{f}_{\sigma mk}(r)$ and $\overset{*}{\boldsymbol{f}}_{\dot{\sigma}\dot{m}\dot{k}}(r^*)$ are radial functions. Taking into account the subgroup T_5, we can rewrite the wavefunctions (100) and (101) in terms of Fourier-type integrals (99) (field operators).

5. Spherical functions of unitary representations of $SO_0(1,4)$

Spherical functions $\mathfrak{M}^l_{mn}(\varphi^q, \theta^q, \psi^q)$, considered in section 4, define matrix elements of non-unitary finite-dimensional representations of the group $SO_0(1,4)$. Following the analogue between $\mathbf{Spin}_+(1,3) \simeq SL(2,\mathbb{C})$ and $\mathbf{Spin}_+(1,4) \simeq Sp(1,1)$, we can define finite-dimensional (spinor) representations of $SO_0(1,4)$ in the space of symmetric polynomials $\mathrm{Sym}_{(k,r)}$ as follows[12]:

$$T_{\mathfrak{q}} q(z,\overline{z}) = (cz+d)^{l_0+l_1-1} \overline{(cz+d)}^{l_0-l_1+1} q\left(\frac{az+b}{cz+d}; \frac{\overline{az+b}}{\overline{cz+d}}\right), \tag{102}$$

where $a, b, c, d \in \mathbb{H}$, $k = l_0 + l_1 - 1$, $r = l_0 - l_1 + 1$, and the pair (l_0, l_1) defines an irreducible representation of $SO_0(1,4)$ in the Diximier–Ström basis [9, 22]:

[12] As is known, any proper Lorentz transformation $\mathfrak{g}$ corresponds to a fractional linear transformation of the complex plane with the matrix $\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in SL(2,\mathbb{C})$ [15]. In its turn, any proper de Sitter transformation $\mathfrak{q}$ can be identified with a fractional linear transformation $w = (az+b)(cz+d)^{-1}$ of the anti-quaternion plane with the matrix $\left[\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right] \in Sp(1,1)$ (about quaternion and anti-quaternion planes and their fractional linear transformations, see [21]).

$$M_3|j',m',q,q;l_0,m,l_1\rangle = m'|j',m',q,q;l_0,m,l_1\rangle,$$

$$M_\pm|j',m',q,q;l_0,m,l_1\rangle = [(j'\mp m')(j'\pm m'+1)]^{\frac{1}{2}}|j',m'+1,q,q;l_0,m,l_1\rangle,$$

$$\begin{aligned}P_3|j',m',q,q;l_0,m,l_1\rangle &= -\alpha(j'+1;q,q)[(j'+1)^2-m^2]^{\frac{1}{2}}|j'+1,m',q,q;l_0,m,l_1\rangle\\ &+\frac{m'(q+1)q}{j'(j'+1)}|j',m',q,q;l_0,m,l_1\rangle\\ &-\alpha(j';q,q)[j'^2-m^2]^{\frac{1}{2}}|j'-1,m',q,q;l_0,m,l_1\rangle,\end{aligned}$$

$$\begin{aligned}&P_\pm|j',m',q,q;l_0,m,l_1\rangle\\ &\quad=\pm\alpha(j'+1;q,q)[(j'\pm m'+1)(j'\pm m'+2)]^{\frac{1}{2}}|j'+1,m'\pm1,q,q;l_0,m,l_1\rangle\\ &\quad+\frac{(q+1)q}{j'(j'+1)}[(j'\mp m')(j'\pm m'+1)]^{\frac{1}{2}}|j',m'\pm1,q,q;l_0,m,l_1\rangle\\ &\quad\mp\alpha(j';q,q)[(j'\mp m')(j'\mp m'-1)]^{\frac{1}{2}}|j'-1,m'\pm1,q,q;l_0,m,l_1\rangle,\end{aligned}$$

$$\begin{aligned}P_0|j',m',q,q;l_0,m,l_1\rangle &= a(q,q;l_0,l_1)[(q+j'+2)(q-j'+1)]^{\frac{1}{2}}|j',m',q+1,q;l_0,m,l_1\rangle\\ &+a(q-1,q;l_0,l_1)[(q+j'+1)(q-j')]^{\frac{1}{2}}|j',m',q-1,q;l_0,m,l_1\rangle\\ &+b(q,q;l_0,l_1)[(j'-q)(j'+q+1)]^{\frac{1}{2}}|j',m',q,q+1;l_0,m,l_1\rangle\\ &+b(q,q-1;l_0,l_1)[(j'+q)(j'-q+1)]^{\frac{1}{2}}|j',m',q,q-1;l_0,m,l_1\rangle,\end{aligned}$$

where $M_\pm = M_1 \pm \mathbf{i}M_2$, $P_\pm = P_1 \pm \mathbf{i}P_2$ and

$$\alpha(j';q,q) = \frac{1}{j'}\left[\frac{(j'^2-q^2)((q+1)^2-j'^2)}{(2j'+1)(2j'-1)}\right]^{\frac{1}{2}},$$

$$a(q,q;l_0,l_1) = \left[\frac{(q-l_0+1)(q+l_0+2)\left(\left(q+\frac{3}{2}\right)^2+l_1^2\right)}{4(2q+1)(q+1)}\right]^{\frac{1}{2}},$$

$$b(q,q;l_0,l_1) = \left[\frac{(l_0-q)(l_0+q+1)\left(\left(q+\frac{1}{2}\right)^2+l_1^2\right)}{4(2q+1)(q+1)}\right]^{\frac{1}{2}}.$$

The relations between the numbers l_0, l_1 and $\sigma, \dot\sigma$ are given by the following formulae:

$$(l_0,l_1) = (\sigma,\sigma+1), \qquad (l_0,l_1) = (-\dot\sigma,\dot\sigma+1),$$

whence it immediately follows that

$$\sigma = \frac{l_0+l_1-1}{2}, \qquad \dot\sigma = \frac{l_0-l_1+1}{2}. \tag{103}$$

In the case of principal series representations of $SO_0(1,4)$ we have[13] $l_1 = -\frac{3}{2}+\mathbf{i}\rho$, $\rho\in\mathbb{R}$. Using formulae (79), (80) and (103), we find that matrix elements of the principal series

[13] This relation is a particular case of the most general formula $l_1 = -\frac{1}{2}(n-1)+\mathbf{i}\rho$ for the principal series representations of $SO_0(1,n)$ [6].

representations of the group $SO_0(1,4)$ have the form

$$\begin{aligned}
\mathfrak{M}_{mn}^{-\frac{3}{2}+\mathbf{i}\rho,l_0}(\mathfrak{q}) &= \exp(-m(\epsilon+\mathbf{i}\varphi+\mathbf{k}\varsigma)-n(\varepsilon+\omega+\mathbf{i}\psi-\mathbf{j}\chi))\\
&\times\sum_{k=-l_0}^{l_0}\sum_{t=-l_0}^{l_0}\mathbf{i}^{m+k-2t}\sqrt{\Gamma(l_0-m+1)\Gamma(l_0+m+1)\Gamma(l_0-t+1)\Gamma(l_0+t+1)}\\
&\times\cos^{2l_0}\frac{\theta}{2}\tan^{m-t}\frac{\theta}{2}\\
&\times\sum_{j=\max(0,t-m)}^{\min(l_0-m,l_0+t)}\frac{\mathbf{i}^{2j}\tan^{2j}\frac{\theta}{2}}{\Gamma(j+1)\Gamma(l_0-m-j+1)\Gamma(l_0+t-j+1)\Gamma(m-t+j+1)}\\
&\times\sqrt{\Gamma(l_0-k+1)\Gamma(l_0+k+1)\Gamma(l_0-t+1)\Gamma(l_0+t+1)}\cos^{2l_0}\frac{\phi}{2}\tan^{k-t}\frac{\phi}{2}\\
&\times\sum_{s=\max(0,t-k)}^{\min(l_0-k,l_0+t)}\frac{\mathbf{i}^{2s}\tan^{2s}\frac{\phi}{2}}{\Gamma(s+1)\Gamma(l_0-k-s+1)\Gamma(l_0+t-s+1)\Gamma(k-t+s+1)}\\
&\times\sqrt{\Gamma\left(-\tfrac{1}{2}+\mathbf{i}\rho-n\right)\Gamma\left(-\tfrac{1}{2}+\mathbf{i}\rho+n\right)\Gamma\left(-\tfrac{1}{2}+\mathbf{i}\rho-k\right)\Gamma\left(-\tfrac{1}{2}+\mathbf{i}\rho+k\right)}\\
&\times\cosh^{-3+2\mathbf{i}\rho}\frac{\tau}{2}\tanh^{n-k}\frac{\tau}{2}\\
&\times\sum_{p=\max(0,k-n)}^{\infty}\frac{\tanh^{2p}\frac{\tau}{2}}{\Gamma(p+1)\Gamma\left(-\frac{1}{2}+\mathbf{i}\rho-n-p\right)\Gamma\left(-\frac{1}{2}+\mathbf{i}\rho+k-p\right)\Gamma(n-k+p+1)}.
\end{aligned}\tag{104}$$

From the latter expression it follows that spherical function $f(\mathfrak{q})$ of the principal series can be defined by means of the function

$$\mathfrak{M}_{mn}^{-\frac{3}{2}+\mathbf{i}\rho,l_0}(\mathfrak{q})=\mathrm{e}^{-m(\epsilon+\mathbf{i}\varphi+\mathbf{k}\varsigma)}\,\mathfrak{Z}_{mn}^{-\frac{3}{2}+\mathbf{i}\rho,l_0}(\cos\theta^q)\,\mathrm{e}^{-n(\varepsilon+\omega+\mathbf{i}\psi-\mathbf{j}\chi)},$$

where

$$\mathfrak{Z}_{mn}^{-\frac{3}{2}+\mathbf{i}\rho,l_0}(\cos\theta^q)=\sum_{k=-l_0}^{l_0}\sum_{t=-l_0}^{l_0}P_{mt}^{l_0}(\cos\theta)\,P_{tk}^{l_0}(\cos\phi)\mathfrak{P}_{kn}^{-\frac{3}{2}+\mathbf{i}\rho}(\cosh\tau).$$

Let us now express the spherical function $\mathfrak{M}_{mn}^{-\frac{3}{2}+\mathbf{i}\rho,l_0}(\mathfrak{q})$ of the principal series representations of $SO_0(1,4)$ via multiple hypergeometric series. Using formulae (104) and (81)–(82), we find

$$\begin{aligned}
\mathfrak{M}_{mn}^{-\frac{3}{2}+\mathbf{i}\rho,l_0}(\mathfrak{q}) &= \exp(-m(\epsilon+\mathbf{i}\varphi+\mathbf{k}\varsigma)-n(\varepsilon+\omega+\mathbf{i}\psi-\mathbf{j}\chi))\\
&\times\sqrt{\frac{\Gamma(l_0+m+1)\Gamma\left(-\frac{1}{2}+\mathbf{i}\rho-n\right)}{\Gamma(l_0-m+1)\Gamma\left(-\frac{1}{2}+\mathbf{i}\rho+n\right)}}\cos^{2l_0}\frac{\theta}{2}\cos^{2l_0}\frac{\phi}{2}\cosh^{-3+2\mathbf{i}\rho}\frac{\tau}{2}\\
&\times\sum_{k=-l_0}^{l_0}\sum_{t=-l_0}^{l_0}\mathbf{i}^{m-k}\sqrt{\frac{\Gamma(l_0-k+1)\Gamma\left(-\frac{1}{2}+\mathbf{i}\rho+k\right)}{\Gamma(l_0+k+1)\Gamma\left(-\frac{1}{2}+\mathbf{i}\rho-k\right)}}\tan^{m-t}\frac{\theta}{2}\tan^{t-k}\frac{\phi}{2}\tanh^{k-n}\frac{\tau}{2}\\
&\times{}_2F_1\left(\begin{matrix}m-l_0,-t-l_0\\ m-t+1\end{matrix}\middle|-\tan^2\frac{\theta}{2}\right){}_2F_1\left(\begin{matrix}t-l_0,-k-l_0\\ t-k+1\end{matrix}\middle|-\tan^2\frac{\phi}{2}\right)\\
&\times{}_2F_1\left(\begin{matrix}k+\frac{3}{2}-\mathbf{i}\rho,-n+\frac{3}{2}-\mathbf{i}\rho\\ k-n+1\end{matrix}\middle|\tanh^2\frac{\tau}{2}\right),\\
&m\geqslant t,\quad t\geqslant k,\quad k\geqslant n;
\end{aligned}\tag{105}$$

$$\mathfrak{M}_{mn}^{-\frac{3}{2}+\mathbf{i}\rho,l_0}(\mathfrak{q}) = \exp(-m(\epsilon+\mathbf{i}\varphi+\mathbf{k}\varsigma)-n(\varepsilon+\omega+\mathbf{i}\psi-\mathbf{j}\chi))$$
$$\times\sqrt{\frac{\Gamma(l_0+m+1)\Gamma\left(-\frac{1}{2}+\mathbf{i}\rho-n\right)}{\Gamma(l-m_0+1)\Gamma\left(-\frac{1}{2}+\mathbf{i}\rho+n\right)}}\cos^{2l_0}\frac{\theta}{2}\cos^{2l_0}\frac{\phi}{2}\cosh^{-3+2\mathbf{i}\rho}\frac{\tau}{2}$$
$$\times\sum_{k=-l_0}^{l_0}\sum_{t=-l_0}^{l_0}\mathbf{i}^{m+k-2t}\sqrt{\frac{\Gamma(l_0-k+1)\Gamma\left(-\frac{1}{2}+\mathbf{i}\rho+k\right)}{\Gamma(l_0+k+1)\Gamma\left(-\frac{1}{2}+\mathbf{i}\rho-k\right)}}\tan^{m-t}\frac{\theta}{2}\tan^{k-t}\frac{\phi}{2}\tanh^{k-n}\frac{\tau}{2}$$
$$\times{}_2F_1\left(\begin{matrix}m-l_0,-t-l_0\\ m-t+1\end{matrix}\middle|-\tan^2\frac{\theta}{2}\right){}_2F_1\left(\begin{matrix}k-l_0,-t-l_0\\ k-t+1\end{matrix}\middle|-\tan^2\frac{\phi}{2}\right)$$
$$\times{}_2F_1\left(\begin{matrix}k+\frac{3}{2}-\mathbf{i}\rho,-n+\frac{3}{2}-\mathbf{i}\rho\\ k-n+1\end{matrix}\middle|\tanh^2\frac{\tau}{2}\right),$$
$$m\geqslant t,\quad k\geqslant t,\quad k\geqslant n; \tag{106}$$

$$\mathfrak{M}_{mn}^{-\frac{3}{2}+\mathbf{i}\rho,l_0}(\mathfrak{q}) = \exp(-m(\epsilon+\mathbf{i}\varphi+\mathbf{k}\varsigma)-n(\varepsilon+\omega+\mathbf{i}\psi-\mathbf{j}\chi))$$
$$\times\sqrt{\frac{\Gamma(l_0-m+1)\Gamma\left(-\frac{1}{2}+\mathbf{i}\rho+n\right)}{\Gamma(l_0+m+1)\Gamma\left(-\frac{1}{2}+\mathbf{i}\rho-n\right)}}\cos^{2l_0}\frac{\theta}{2}\cos^{2l_0}\frac{\phi}{2}\cosh^{-3+3\mathbf{i}\rho}\frac{\tau}{2}$$
$$\times\sum_{k=-l_0}^{l_0}\sum_{t=-l_0}^{l_0}\mathbf{i}^{k-m}\sqrt{\frac{\Gamma(l_0+k+1)\Gamma\left(-\frac{1}{2}+\mathbf{i}\rho-k\right)}{\Gamma(l_0-k+1)\Gamma\left(-\frac{1}{2}+\mathbf{i}\rho+k\right)}}\tan^{t-m}\frac{\theta}{2}\tan^{k-t}\frac{\phi}{2}\tanh^{n-k}\frac{\tau}{2}$$
$$\times{}_2F_1\left(\begin{matrix}t-l_0,-m-l_0\\ t-m+1\end{matrix}\middle|-\tan^2\frac{\theta}{2}\right){}_2F_1\left(\begin{matrix}k-l_0,-t-l_0\\ k-t+1\end{matrix}\middle|-\tan^2\frac{\phi}{2}\right)$$
$$\times{}_2F_1\left(\begin{matrix}n+\frac{3}{2}-\mathbf{i}\rho,-k+\frac{3}{2}-\mathbf{i}\rho\\ n-k+1\end{matrix}\middle|\tanh^2\frac{\tau}{2}\right),$$
$$t\geqslant m,\quad k\geqslant t,\quad n\geqslant k; \tag{107}$$

$$\mathfrak{M}_{mn}^{-\frac{3}{2}+\mathbf{i}\rho,l_0}(\mathfrak{q}) = \exp(-m(\epsilon+\mathbf{i}\varphi+\mathbf{k}\varsigma)-n(\varepsilon+\omega+\mathbf{i}\psi-\mathbf{j}\chi))$$
$$\times\sqrt{\frac{\Gamma(l_0-m+1)\Gamma\left(-\frac{1}{2}+\mathbf{i}\rho+n\right)}{\Gamma(l_0+m+1)\Gamma\left(-\frac{1}{2}+\mathbf{i}\rho-n\right)}}\cos^{2l_0}\frac{\theta}{2}\cos^{2l_0}\frac{\phi}{2}\cosh^{-3+2\mathbf{i}\rho}\frac{\tau}{2}$$
$$\times\sum_{k=-l_0}^{l_0}\sum_{t=-l_0}^{l_0}\mathbf{i}^{2t-m-k}\sqrt{\frac{\Gamma(l_0+k+1)\Gamma\left(-\frac{1}{2}+\mathbf{i}\rho-k\right)}{\Gamma(l_0-k+1)\Gamma\left(-\frac{1}{2}+\mathbf{i}\rho+k\right)}}\tan^{t-m}\frac{\theta}{2}\tan^{t-k}\frac{\phi}{2}\tanh^{n-k}\frac{\tau}{2}$$
$$\times{}_2F_1\left(\begin{matrix}t-l_0,-m-l_0\\ t-m+1\end{matrix}\middle|-\tan^2\frac{\theta}{2}\right){}_2F_1\left(\begin{matrix}t-l_0,-k-l_0\\ t-k+1\end{matrix}\middle|-\tan^2\frac{\phi}{2}\right)$$
$$\times{}_2F_1\left(\begin{matrix}n+\frac{3}{2}-\mathbf{i}\rho,-k+\frac{3}{2}-\mathbf{i}\rho\\ n-k+1\end{matrix}\middle|\tanh^2\frac{\tau}{2}\right),$$
$$t\geqslant m,\quad t\geqslant k,\quad n\geqslant k; \tag{108}$$

$$\mathfrak{M}_{mn}^{-\frac{3}{2}+\mathbf{i}\rho,l_0}(\mathfrak{q}) = \exp(-m(\epsilon+\mathbf{i}\varphi+\mathbf{k}\varsigma)-n(\varepsilon+\omega+\mathbf{i}\psi-\mathbf{j}\chi))$$
$$\times\sqrt{\frac{\Gamma(l_0-m+1)\Gamma\left(-\frac{1}{2}+\mathbf{i}\rho-n\right)}{\Gamma(l_0+m+1)\Gamma\left(-\frac{1}{2}+\mathbf{i}\rho+n\right)}}\cos^{2l_0}\frac{\theta}{2}\cos^{2l_0}\frac{\phi}{2}\cosh^{-3+2\mathbf{i}\rho}\frac{\tau}{2}$$

$$\times \sum_{k=-l_0}^{l_0}\sum_{t=-l_0}^{l_0} \mathbf{i}^{k-m}\sqrt{\frac{\Gamma(l_0+k+1)\Gamma\left(-\frac{1}{2}+\mathbf{i}\rho+k\right)}{\Gamma(l_0-k+1)\Gamma\left(-\frac{1}{2}+\mathbf{i}\rho-k\right)}}\tan^{t-m}\frac{\theta}{2}\tan^{k-t}\frac{\phi}{2}\tanh^{k-n}\frac{\tau}{2}$$
$$\times {}_2F_1\left(\begin{matrix}t-l_0,-m-l_0\\ t-m+1\end{matrix}\,\middle|\,-\tan^2\frac{\theta}{2}\right){}_2F_1\left(\begin{matrix}k-l_0,-t-l_0\\ k-t+1\end{matrix}\,\middle|\,-\tan^2\frac{\phi}{2}\right)$$
$$\times {}_2F_1\left(\begin{matrix}k+\frac{3}{2}-\mathbf{i}\rho,-n+\frac{3}{2}-\mathbf{i}\rho\\ k-n+1\end{matrix}\,\middle|\,\tanh^2\frac{\tau}{2}\right),$$
$$t\geqslant m,\quad k\geqslant t,\quad k\geqslant n; \tag{109}$$

$$\mathfrak{M}^{-\frac{3}{2}+\mathbf{i}\rho,l_0}_{mn}(\mathfrak{q})=\exp(-m(\epsilon+\mathbf{i}\varphi+\mathbf{k}\varsigma)-n(\varepsilon+\omega+\mathbf{i}\psi-\mathbf{j}\chi))$$
$$\times\sqrt{\frac{\Gamma(l_0-m+1)\Gamma\left(-\frac{1}{2}+\mathbf{i}\rho-n\right)}{\Gamma(l_0+m+1)\Gamma\left(-\frac{1}{2}+\mathbf{i}\rho+n\right)}}\cos^{2l_0}\frac{\theta}{2}\cos^{2l_0}\frac{\phi}{2}\cosh^{-3+2\mathbf{i}\rho}\frac{\tau}{2}$$
$$\times\sum_{k=-l_0}^{l_0}\sum_{t=-l_0}^{l_0}\mathbf{i}^{2t-m-k}\sqrt{\frac{\Gamma(l_0+k+1)\Gamma\left(-\frac{1}{2}+\mathbf{i}\rho+k\right)}{\Gamma(l_0-k+1)\Gamma\left(-\frac{1}{2}+\mathbf{i}\rho-k\right)}}\tan^{t-m}\frac{\theta}{2}\tan^{t-k}\frac{\phi}{2}\tanh^{k-n}\frac{\tau}{2}$$
$$\times {}_2F_1\left(\begin{matrix}t-l_0,-m-l_0\\ t-m+1\end{matrix}\,\middle|\,-\tan^2\frac{\theta}{2}\right){}_2F_1\left(\begin{matrix}t-l_0,-k-l_0\\ t-k+1\end{matrix}\,\middle|\,-\tan^2\frac{\phi}{2}\right)$$
$$\times {}_2F_1\left(\begin{matrix}k+\frac{3}{2}-\mathbf{i}\rho,-n+\frac{3}{2}-\mathbf{i}\rho\\ k-n+1\end{matrix}\,\middle|\,\tanh^2\frac{\tau}{2}\right),$$
$$t\geqslant m,\quad t\geqslant k,\quad k\geqslant n; \tag{110}$$

$$\mathfrak{M}^{-\frac{3}{2}+\mathbf{i}\rho,l_0}_{mn}(\mathfrak{q})=\exp(-m(\epsilon+\mathbf{i}\varphi+\mathbf{k}\varsigma)-n(\varepsilon+\omega+\mathbf{i}\psi-\mathbf{j}\chi))$$
$$\times\sqrt{\frac{\Gamma(l_0+m+1)\Gamma\left(-\frac{1}{2}+\mathbf{i}\rho+n\right)}{\Gamma(l_0-m+1)\Gamma\left(-\frac{1}{2}+\mathbf{i}\rho-n\right)}}\cos^{2l_0}\frac{\theta}{2}\cos^{2l_0}\frac{\phi}{2}\cosh^{-3+2\mathbf{i}\rho}\frac{\tau}{2}$$
$$\times\sum_{k=-l_0}^{l_0}\sum_{t=-l_0}^{l_0}\mathbf{i}^{m-k}\sqrt{\frac{\Gamma(l_0-k+1)\Gamma\left(-\frac{1}{2}+\mathbf{i}\rho-k\right)}{\Gamma(l_0+k+1)\Gamma\left(-\frac{1}{2}+\mathbf{i}\rho+k\right)}}\tan^{m-t}\frac{\theta}{2}\tan^{t-k}\frac{\phi}{2}\tanh^{n-k}\frac{\tau}{2}$$
$$\times {}_2F_1\left(\begin{matrix}m-l_0,-t-l_0\\ m-t+1\end{matrix}\,\middle|\,-\tan^2\frac{\theta}{2}\right){}_2F_1\left(\begin{matrix}t-l_0,-k-l_0\\ t-k+1\end{matrix}\,\middle|\,-\tan^2\frac{\phi}{2}\right)$$
$$\times {}_2F_1\left(\begin{matrix}n+\frac{3}{2}-\mathbf{i}\rho,-k+\frac{3}{2}-\mathbf{i}\rho\\ n-k+1\end{matrix}\,\middle|\,\tanh^2\frac{\tau}{2}\right),$$
$$m\geqslant t,\quad t\geqslant k,\quad n\geqslant k; \tag{111}$$

$$\mathfrak{M}^{-\frac{3}{2}+\mathbf{i}\rho,l_0}_{mn}(\mathfrak{q})=\exp(-m(\epsilon+\mathbf{i}\varphi+\mathbf{k}\varsigma)-n(\varepsilon+\omega+\mathbf{i}\psi-\mathbf{j}\chi))$$
$$\times\sqrt{\frac{\Gamma(l_0+m+1)\Gamma\left(-\frac{1}{2}+\mathbf{i}\rho+n\right)}{\Gamma(l_0-m+1)\Gamma\left(-\frac{1}{2}+\mathbf{i}\rho-n\right)}}\cos^{2l_0}\frac{\theta}{2}\cos^{2l_0}\frac{\phi}{2}\cosh^{-3+2\mathbf{i}\rho}\frac{\tau}{2}$$
$$\times\sum_{k=-l_0}^{l_0}\sum_{t=-l_0}^{l_0}\mathbf{i}^{m+k-2t}\sqrt{\frac{\Gamma(l_0-k+1)\Gamma\left(-\frac{1}{2}+\mathbf{i}\rho-k\right)}{\Gamma(l_0+k+1)\Gamma\left(-\frac{1}{2}+\mathbf{i}\rho+k\right)}}\tan^{m-t}\frac{\theta}{2}\tan^{k-t}\frac{\phi}{2}\tanh^{n-k}\frac{\tau}{2}$$

$$\times\ {}_2F_1\left(\begin{matrix} m-l_0,\,-t-l_0 \\ m-t+1 \end{matrix}\,\middle|\,-\tan^2\frac{\theta}{2}\right){}_2F_1\left(\begin{matrix} k-l_0,\,-t-l_0 \\ k-t+1 \end{matrix}\,\middle|\,-\tan^2\frac{\phi}{2}\right)$$
$$\times\ {}_2F_1\left(\begin{matrix} n+\frac{3}{2}-\mathbf{i}\rho,\,-k+\frac{3}{2}-\mathbf{i}\rho \\ n-k+1 \end{matrix}\,\middle|\,\tanh^2\frac{\tau}{2}\right),$$
$$m\geqslant t,\quad k\geqslant t,\quad n\geqslant k. \tag{112}$$

Spherical functions of the second type $f(\varphi^q,\theta^q)=\mathfrak{M}^{m}_{-\frac{3}{2}+\mathbf{i}\rho,l_0}(\varphi^q,\theta^q,0)$ of the principal series are defined as

$$\mathfrak{M}^{m}_{-\frac{3}{2}+\mathbf{i}\rho,l_0}(\varphi^q,\theta^q,0)=\mathrm{e}^{-\mathbf{i}m\varphi^q}\,\mathfrak{Z}^{m}_{-\frac{3}{2}+\mathbf{i}\rho,l_0}(\cos\theta^q),$$

where

$$\mathfrak{Z}^{m}_{-\frac{3}{2}+\mathbf{i}\rho,l_0}(\cos\theta^q)=\sum_{k=-l_0}^{l_0}\sum_{t=-l_0}^{l_0}P^{l_0}_{mt}(\cos\theta)\,P^{l_0}_{tk}(\cos\phi)\mathfrak{P}^{k}_{-\frac{3}{2}+\mathbf{i}\rho}(\cosh\tau).$$

Hypergeometric-type formulae for the functions $f(\varphi^q,\theta^q)$ follow directly from (105) to (112) at $n=0$.

Spherical functions of the third type $f(\epsilon,\tau,\varepsilon,\omega)=\mathfrak{M}^{-\frac{3}{2}+\mathbf{i}\rho}_{mn}(\epsilon,\tau,\varepsilon,\omega)$ for the principal series representations have the form

$$\mathfrak{M}^{-\frac{3}{2}+\mathbf{i}\rho}_{mn}(\epsilon,\tau,\varepsilon,\omega)=\mathrm{e}^{-m\epsilon}\,\mathfrak{P}^{-\frac{3}{2}+\mathbf{i}\rho}_{mn}(\cosh\tau)\,\mathrm{e}^{-n(\varepsilon+\omega)}.$$

The hypergeometric-type formulae are

$$\mathfrak{M}^{-\frac{3}{2}+\mathbf{i}\rho}_{mn}(\epsilon,\tau,\varepsilon,\omega)=\mathrm{e}^{-m\epsilon-n(\varepsilon+\omega)}\sqrt{\frac{\Gamma\left(\mathbf{i}\rho+m-\frac{1}{2}\right)\Gamma\left(\mathbf{i}\rho-n-\frac{1}{2}\right)}{\Gamma(\mathbf{i}\rho-m-\frac{1}{2})\Gamma(\mathbf{i}\rho+n-\frac{1}{2})}}\cosh^{-3+2\mathbf{i}\rho}\frac{\tau}{2}$$
$$\times\tanh^{m-n}\frac{\tau}{2}\,{}_2F_1\left(\begin{matrix} m-\mathbf{i}\rho-\frac{1}{2},\,-n-\mathbf{i}\rho-\frac{1}{2} \\ m-n+1 \end{matrix}\,\middle|\,\tanh^2\frac{\tau}{2}\right),\qquad m\geqslant n;$$

$$\mathfrak{M}^{-\frac{3}{2}+\mathbf{i}\rho}_{mn}(\epsilon,\tau,\varepsilon,\omega)=\mathrm{e}^{-m\epsilon-n(\varepsilon+\omega)}\sqrt{\frac{\Gamma(\mathbf{i}\rho+n-\frac{1}{2})\Gamma(\mathbf{i}\rho-m-\frac{1}{2})}{\Gamma\left(\mathbf{i}\rho-n-\frac{1}{2}\right)\Gamma\left(\mathbf{i}\rho+m-\frac{1}{2}\right)}}\cosh^{-3+2\mathbf{i}\rho}\frac{\tau}{2}$$
$$\times\tanh^{n-m}\frac{\tau}{2}\,{}_2F_1\left(\begin{matrix} n-\mathbf{i}\rho-\frac{1}{2},\,-m-\mathbf{i}\rho-\frac{1}{2} \\ n-m+1 \end{matrix}\,\middle|\,\tanh^2\frac{\tau}{2}\right),\qquad n\geqslant m.$$

In like manner we can define conjugated spherical functions $f(\mathfrak{q})=\mathfrak{M}^{-\frac{3}{2}-\mathbf{i}\rho,l_0}_{\dot m\dot n}(\mathfrak{q})$, $f(\dot\varphi^q,\dot\theta^q)=\mathfrak{M}^{\dot m}_{-\frac{3}{2}-\mathbf{i}\rho,l_0}(\dot\varphi^q,\dot\theta^q,0)$ and $\dot f(\epsilon,\tau,\varepsilon,\omega)=\mathfrak{M}^{-\frac{3}{2}-\mathbf{i}\rho}_{\dot m\dot n}(\epsilon,\tau,\varepsilon,\omega)$, since a conjugated representation of $SO_0(1,4)$ is defined by the pair $\pm(l_0,-l_1)$.

References

[1] Agamaliev A K, Atakishiev N M and Verdiev I A 1969 Invariant expansion of solutions of relativistic equations *Yad. Fiz.* **9** 201–11

[2] Allen B 1985 Vacuum states in de Sitter space *Phys. Rev.* D **32** 3136

[3] Appell P and Kampé de Fériet M J 1926 *Fonctions hypérgeométriques et hypersphéques. Polynomes d'Hermite* (Paris: Gauthier-Villars)

[4] Bander M and Itzykson C 1966 Group theory and the hydrogen atom *Rev. Mod. Phys.* **38** 330–58

[5] Bisiacchi G, Budini P and Calucci G 1968 Majorana equations for composite systems *Phys. Rev.* **172** 1508–15

[6] Boyer C P 1971 Matrix elements for the most degenerate continuous principal series of representations of $SO(p, 1)$ *J. Math. Phys.* **12** 1599–603
[7] Bros J and Moschella U 1996 Two-point functions and quantum fields in de Sitter universe *Rev. Math. Phys.* **8** 327–92
[8] Dirac P A M 1935 The electron wave equation in de Sitter space *Ann. Math.* **36** 657
[9] Dixmier J 1961 Représentations intégrables du groupe de De Sitter *Bull. Soc. Math. France* **89** 9–41
[10] Eades J and Hartmann F J 1999 Forty years of antiprotons *Rev. Mod. Phys.* **71** 373–419
[11] Exton H 1976 *Multiple Hypergeometric Functions and Applications* (Chicester: Ellis Horwood)
[12] Fock V A 1935 Zur Theorie des Wasserstoffatoms *Z. Phys.* **98** 145–54
[13] Gel'fand I M and Yaglom A M 1948 General relativistic-invariant equations and infinite-dimensional representations of the Lorentz group *Zh. Ehsp. Teor. Fiz.* **18** 703–33
[14] Gel'fand I M, Graev M I and Ya Vilenkin N 1962 *Integral Geometry and Related Questions of Group Representations* (Moskow: Fizmatgiz) (in Russian)
[15] Gel'fand I M, Minlos R A and Shapiro Z Ya 1963 *Representations of the Rotation and Lorentz Groups and their Applications* (Oxford: Pergamon)
[16] Helgason S 1978 *Differential Geometry, Lie Groups and Symmetric Spaces* (New York: Academic)
[17] Karoubi M 1979 *K-Theory. An Introduction* (Berlin: Springer)
[18] Klimyk A U and Kachurik I I 1986 *Calculating Methods in Group Representations Theory* (Kiev: Visha Shkola) (in Russian)
[19] Moradi S, Rouhani S and Takook M V 2005 Discrete symmetries for spinor field in de Sitter space *Phys. Lett.* B **613** 74–82
[20] Naimark M A 1964 *Linear Representations of the Lorentz Group* (London: Pergamon)
[21] Rozenfel'd B A 1955 *Non-Euclidean Geometries* (Moscow: Gostehizdat) (in Russian)
[22] Ström S 1969 On the decomposition of a unitary representation of (1+4) de Sitter group with respect to representations of the Lorentz group *Ark. Fys.* **40** 1–33
[23] Takahashi R 1963 Sur les représentations unitaries des groupes de Lorentz généralisés *Bull. Soc. Math. France* **91** 289–433
[24] Varlamov V V 2003 General solutions of relativistic wave equations *Int. J. Theor. Phys.* **42** 583–633
[25] Varlamov V V 2004 Relativistic wavefunctions on the Poincaré group *J. Phys. A: Math. Gen.* **37** 5467–76
[26] Varlamov V V 2005 Maxwell field on the Poincaré group *Int. J. Mod. Phys.* A **20** 4095–112
[27] Varlamov V V 2004 Universal coverings of orthogonal groups Adv. *Appl. Clifford Algebras* **14** 81–168
[28] Varlamov V V 2005 General solutions of relativistic wave equations: II. Arbitrary spin chains *Preprint* math-ph/0503058 (*Int. J. Theor. Phys.* at press)
[29] Varlamov V V 2005 CPT groups for spinor field in de Sitter space *Phys. Lett.* B **631** 187–91
[30] Varlamov V V 2006 Relativistic spherical functions on the Lorentz group *J. Phys. A: Math. Gen.* **39** 805–22
[31] Ya Vilenkin N 1968 *Special Functions and the Theory of Group Representations* (Providence, RI: American Mathematical Society)
[32] Ya Vilenkin N and Klimyk A U 1991–1993 *Representations of Lie Groups and Special Functions* vols 1–3 (Dordrecht: Kluwer)